# Math 246B Presentation

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## 1. DeRham Cohomology

### 1.1. Definition of DeRham Cohomology.

Let $M$ be a smooth $n$-dimensional manifold. Let $\Omega^k(M)$ be the vector space of differential $k$-forms on $M$. Recall that the wedge product is a map $\wedge : \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$ and define

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M).$$

Together with the wedge product $\Omega^*(M)$ is an associative, anticommutative, graded algebra.

Now let's define another map called the exterior derivative:

**Theorem 1** (The Exterior Derivative)**.** *Let $M$ be a smooth manifold. For each integer $k \geq 0$ there are unique linear maps*

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

*satisfying:*

(1) *If $f : M \to \mathbb{R}$ is smooth (i.e. $f \in \Omega^0(M)$), then $df$ is the differential of $f$, defined by*

$$df(X) = X(f)$$

*where $X$ is a smooth vector field on $M$.*

(2) *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

(3) $d^2 = 0$.

One more definition before interesting stuff!

**Definition 1** (Closed and Exact Forms)**.** *Let $\omega \in \Omega^k(M)$. We say that $\omega$ is a* **closed** *$k$-form if $d\omega = 0$. If $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$ then we say that $\omega$ is an* **exact** *$k$-form.*

Notice that every exact $k$-form is closed.

Notice that $(\Omega^*(M), d)$ forms the following chain complex of vector spaces:

$$\cdots \longrightarrow 0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0 \longrightarrow \cdots$$

(recall that $M$ is an $n$-manifold and that there are no $k$-forms on an $n$-manifold where $k > n$). Observe for a moment the map $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$. Notice the following two things:

$$\mathcal{C}^k(M) = \text{ Ker } d = \{\text{closed } k\text{-forms on } M\}$$

and

$$\mathcal{E}^{k+1}(M) = \text{ Im } d = \{\text{exact } k+1\text{-forms on } M\}.$$

From the comment above that every exact $k$-form is a closed $k$-form, we see that $\mathcal{E}^k(M) \subset \mathcal{C}^k(M)$. Since $(\Omega^*(M), d)$ is a chain complex, we can speak of the cohomology (since $d$ is a degree increasing map) of this complex:

**Definition 2** (deRham Cohomology)**.** *Let $(\Omega^*(M), d)$, $\mathcal{C}^k(M)$, and $\mathcal{E}^k(M)$ be defined as above. Define the $q^{th}$* **deRham cohomology vector space of** *$M$ by*

$$H^q_{dR}(M) = \mathcal{C}^q(M)/\mathcal{E}^q(M) = \{\textit{closed } q\textit{-forms on } M\}/\{\textit{exact } q\textit{-forms on } M\}.$$

## 1.2. Examples of DeRham Cohomology Spaces.

**Example 1** (Zero-Dimensional deRham Cohomology). *$M$ a connected smooth manifold,*

$$H^0_{dR}(M) = \{\textit{constant functions } f : M \to \mathbb{R}\} \cong \mathbb{R}$$

**Example 2** (deRham Cohomology of Zero-Manifolds). *$M$ a 0-manifold, then* $\dim H^0_{dR}(M) = o(M)$ *and* $H^q_{dR}(M) = 0$ *for all $q \geq 1$.*

*Computation.* Since $M$ is a 0-manifold it is a discrete set, hence we may think of $M = \bigsqcup_{m \in M} \{m\}$. Then, the inclusion maps $\iota_m : \{m\} \hookrightarrow M$ induce an isomorphism:

$$H^0_{dR}(M) = H^0_{dR}\left(\bigsqcup_{m\in M}\{m\}\right) \cong \prod_{m\in M} H^0_{dR}(\{m\}) \cong \prod_{m \in M} \mathbb{R}$$

Hence $\dim H^0_{dR}(M) = o(M)$. Moreover, since there are no $q$-forms on $M$ for $q \geq 1$, it is impossible for $H^q_{dR}(M)$ to be nonzero for anything but $q = 0$. □

**Theorem 2** (The Poincaré Lemma). *Let $U$ be a star-shaped open subset of $\mathbb{R}^n$. Then $H^q_{dR}(U) = 0$ for $q \geq 1$.*

*Proof.* Suppose that $q \geq 1$ and that $U$ is star-shaped with respect to a point $p \in U$. Then $U$ is a contractible space. By the homotopy invariance of the deRham cohomology, $H^q_{dR}(U) \cong H^q_{dR}(\{p\})$. The previous example shows that $H^q_{dR}(U) = 0$. □

**Corollary 1.** *For all $q \geq 1$, $H^q_{dR}(\mathbb{R}^n) = 0$.*

**Example 3** (deRham Cohomology of Spheres). *For $n \geq 1$, the deRham cohomology groups of $S^n$ are:*

$$H^q_{dR}(S^n) = \begin{cases} \mathbb{R}, & q = 0, n \\ 0, & \textit{otherwise} \end{cases}$$

## 1.3. The Cup Product.

**Proposition 1.** *Let $M$ be a smooth $n$-manifold and let $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$ be closed forms. Then the deRham cohomology class of $\omega \wedge \eta$ depends only on the deRham cohomology classes of $\omega$ and $\eta$.*

**Corollary 2.** *There is a well-defined bilinear map:*

$$\smile : H^p_{dR}(M) \times H^q_{dR}(M) \longrightarrow H^{p+q}_{dR}(M)$$

*given by*

$$[\omega] \smile [\eta] = [\omega \wedge \eta].$$

The map in the corollary gives us the deRham cohomology algebra.

## 1.4. DeRham Cohomology and Orientability.

For those of you who don't remember this fundamental theorem:

**Theorem 3** (Stokes' Theorem). *Let $M$ be a smooth, oriented $n$-manifold with boundary, and let $\omega$ be a smooth, compactly supported, $(n-1)$-form on $M$. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

We can also detect orientability using the top cohomology as follows:

First we define the integration map: $I : \Omega^p(M) \to \mathbb{R}$ by $I(\omega) = \int_M \omega$. Clearly $I$ is a linear map. Because the integral of any exact differential form is zero (by Stokes' theorem), we get that $I$ decends to a linear map on $H^q_{dR}(M)$, i.e., we now have a linear $I : H^q_{dR}(M) \to \mathbb{R}$. Necessarily we have that $\Omega^n(M) = \mathcal{C}^n(M)$ for an $n$-manifold $M$. Recall that any orientable $n$-manifold has a nonvanishing $n$-form.

**Proposition 2** (Top Cohomology and Orientability)**.** *Let $M$ be a compact, connected, smooth $n$-manifold.*

(1) *If $M$ is orientable, the map $I : H^n_{dR}(M) \to \mathbb{R}$ is an isomorphism.*
(2) *If $M$ is nonorientable, then $H^n_{dR}(M) = 0$.*

## 2. Smooth Singular Homology

**Definition 3.** *If $M$ is a smooth manifold, and $\Delta^q$ is the standard $q$-simplex, define a* **smooth $q$-simplex in** *$M$ to be a smooth map $\sigma : \Delta^q \to M$. (Smooth in the sense that at every point there is a smooth extension to an open neighborhood of the point.) Denote the subset of $C_q(M)$ generated by smooth $q$-simplices by $C^\infty_q(M)$ and call it the* $q^{th}$**-smooth chain group**. *The elements of these groups are called* **smooth chains**. *Because of this, we may define the $q^{th}$* **smooth singular homology group of** *$M$ to be*

$$H^\infty_q(M) = \textit{ Ker}\,\{\partial : C^\infty_q(M) \to C^\infty_{q-1}(M)\}/\textit{ Im}\,\{\partial : C^\infty_{q+1}(M) \to C^\infty_q(M)\}.$$

Since the inclusion map $\iota : C^\infty_q(M) \hookrightarrow C_q(M)$ commutes with $\partial$, we get an induced map on homology $\iota_* : H^\infty_q(M) \to H_q(M)$ given by $\iota_*([c]) = [\iota(c)]$. In fact:

**Theorem 4.** *For any smooth manifold $M$, the map $\iota_* : H^\infty_q(M) \to H_q(M)$ induced by inclusion is an isomorphism.*

*Proof.* See John Lee's "Introduction to Smooth Manifolds" pgs.417-424. The basic idea of the proof is to get a homotopy between a smooth $q$-simplex and a regular $q$-simplex using the Weierstra$\beta$ smooth approximation theorem. $\square$

Curiously, it works out, much to our convienence, that $H^q(M;\mathbb{R}) \cong \mathrm{Hom}(H_q(M),\mathbb{R}) \cong \mathrm{Hom}(H^\infty_q(M),\mathbb{R})$.

## 3. The DeRham Theorem

For a smooth manifold $M$, $\omega \in \Omega^q(M)$ and $\sigma \in C^\infty_q(M)$, define the integral of $\omega$ over $\sigma$ to be

$$\int_\sigma \omega = \int_{\Delta^q} \sigma^*\omega.$$

It is, in fact, because of this that we want to look as smooth simplices in $M$ since we can only pull back a differential form under a smooth map.

**Theorem 5** (Stokes' Theorem for Chains)**.** *If $c$ is a smooth $q$-chain in a smooth manifold $M$, and $\omega$ is a smooth $(q-1)$-form on $M$, then*

$$\int_{\partial c} \omega = \int_c d\omega.$$

This theorem furnishes a linear map:

$$\eta : H^q_{dR}(M) \to H^q(M;\mathbb{R})$$

called the **deRham homomorphism** and is defined by:

$$\eta([\omega])[c] = \int_{\tilde{c}} \omega$$

where $[\omega] \in H^q_{dR}(M)$, $[c] \in H_q(M) \cong H^\infty_q(M)$, and $\tilde{c}$ is any representative of $[c]$.

The deRham homomorphism is natural, that is, given a smooth map $F : M \to N$ of manifolds, the following diagram commutes:

$$\begin{array}{ccc} H^q_{dR}(N) & \xrightarrow{F^*} & H^q_{dR}(M) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\ H^q(N;\mathbb{R}) & \xrightarrow{F^*} & H^q(M;\mathbb{R}) \end{array}$$

Before we can prove the main attraction, we need three definitions:

**Definition 4** (deRham Manifold)**.** *We say a smooth manifold $M$ is a* **deRham manifold** *if the map $\eta : H^q_{dR}(M) \to H^q(M;\mathbb{R})$ is an isomorpism for each $q$. (This definition is diffeomorphism invariant by the naturality of $\eta$.)*

**Definition 5** (deRham Cover)**.** *If $M$ is a smooth manifold, then an open cover $\{U_i\}_{i \in I}$ is called a* **deRham cover** *if each $U_i$ is a deRham manifold. A deRham cover that is also a basis for $M$ is called a* **deRham basis** *of $M$.*

**Theorem 6** (The deRham Theorem)**.** *For every smooth manifold $M$ and every $q \in \mathbb{N}_0$, the deRham homomorphism $\eta : H^q_{dR}(M) \to H^q(M;\mathbb{R})$ is an isomorpism.*

*Idea of Proof.* This proof will be broken into 6 steps:

(1) If $\{M_j\}_{j \in J}$ is any countable collection of deRham manifolds, then their disjoint union is deRham.
Use that fact that if $M = \bigsqcup_{j \in J} M_j$, the inclusion maps $\iota_j : M_j \hookrightarrow M$ induces an isomorphism from the cohomology of the disjoint union to the product of the cohomologies of each $M_j$ (both deRham and singular cohomology). Then naturality preserves these isomorphisms.

(2) Every convex open subset of $\mathbb{R}^n$ is deRham.
Use the Poincaré lemma to get isomorphisms for $q > 0$ and just show that both zeroth cohomologies are one dimensional and that $\eta$ is not the zero map here.

(3) If $M$ has a finite deRham cover, then $M$ is deRham.
Use induction on the number of open sets. Use the Mayer-Vietoris sequence on both deRham and singular cohomology and link them with $\eta$ which says that you get commutative squares, then use the five lemma. For the case of a cover with 2 sets, Mayer-Vietoris gives:

$$\begin{array}{ccccccccc}
H^{q-1}_{dR}(U) \oplus H^{q-1}_{dR}(V) & \longrightarrow & H^{q-1}_{dR}(U \cap V) & \longrightarrow & H^q_{dR}(M) & \longrightarrow & H^q_{dR}(U) \oplus H^q_{dR}(V) & \longrightarrow & H^q_{dR}(U \cap V) \\
\downarrow \eta & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta \\
H^{q-1}(U;\mathbb{R}) \oplus H^{q-1}(V;\mathbb{R}) & \longrightarrow & H^{q-1}(U \cap V;\mathbb{R}) & \longrightarrow & H^q(M;\mathbb{R}) & \longrightarrow & H^q(U;\mathbb{R}) \oplus H^q(V;\mathbb{R}) & \longrightarrow & H^q(U \cap V;\mathbb{R})
\end{array}$$

and by naturality of $\eta$ all of these squares commute. By assumption the 1st, 2nd, 4th, and 5th $\eta$'s are isomorphisms, so by the five lemma, since the Mayer-Vietoris sequences are exact, the 3rd must also be an isomorphism.

(4) If $M$ has a deRham basis, then $M$ is deRham.
Use an exhaustion function (a continuous function $f : M \to \mathbb{R}$ such that $M_c = \{m \in M \mid f(m) \le c\}$ is compact, in fact, e
to construct a basis and use steps 1 and 3 to show it is a deRham basis and that $M$ is deRham.

(5) Any open subset of $\mathbb{R}^n$ is deRham.
If $U \subset \mathbb{R}^n$ is open, then it has a basis consisting of open balls, which are convex as are their intersections. Thus $U$ is deRham by steps 2 and 4.

(6) Every smooth manifold is deRham.
Every smooth manifold has a basis of coordinate charts. Each coordinate chart is diffeomorphic to an open subset of $\mathbb{R}^n$ (and their intersections are too). Thus every smooth manifold has a deRham basis by step 5, and hence is deRham by step 4.

□

To conclude, let's answer the question of why anyone should care about this:

Obviously this theorem establishes a connection between the topological and analytic properties of a smooth manifold. For example, if one knows something about the topology of the manifold, you could infer things about differential equations such as $d\omega = \eta$ on $M$; and conversely.