Math 60380 - Basic Complex Analysis II

Final Presentation: J-holomorphic Curves and Applications

Edward Burkard

1. Introduction and Definitions

1.1. **Almost Complex Manifolds.** We begin with a even dimensional manifold V^{2n}. From this, we can form the tangent bundle TV. From the tangent bundle, we can construct a new vector bundle over V, the *endomorphism bundle*, $\operatorname{End}(TV)$, whose fiber at each point $x \in V$ is the space of endomorphisms of T_xV.

Definition 1 (Almost Complex Structure)**.** *An* almost complex structure *on V is a section J of* $\operatorname{End}(TV)$ *such that $J^2 = -\operatorname{id}$.*

Remark. *An almost complex structure J is a* complex structure *if it is integrable, i.e., the Nijenhuis tensor N_J is zero, where*

$$N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

for vector fields X and Y.

A pair (V, J), where J is an almost complex structure on V is called an *almost complex manifold.*

1.2. **J-holomorphic Curves.** Fix a Riemann surface (Σ, j), where j is a complex structure on Σ. A smooth function $u : (\Sigma, j) \to (V, J)$ is called a *J-holomorphic curve* (more precisely a *(j, J)-holomorphic curve*) if du is complex linear with respect to j and J, i.e.,

$$J \circ du = du \circ j.$$

Since j will be fixed throughout our discussions, we will often neglect to mention it, except if required in equations. By composing with J on the left, we can rewrite this equation as

$$du + J \circ du \circ j = 0.$$

The complex antilinear part of du (with respect to J) is

$$\bar{\partial}_J u := \frac{1}{2}(du + J \circ du \circ j),$$

so we can reformulate the definition of a J-holomorphic curve to be the smooth functions which are a solution of the equation

$$\bar{\partial}_J u = 0.$$

This is the analogue of the Cauchy-Riemann equations for J-holomorphic curves. Let's see that this makes sense with our usual notion of holomorphic on $\mathbb{C}^n$:

Let's first start with passing to local coordinates on Σ. We can work in a chart $\phi_\alpha : U_\alpha \to \mathbb{C}$ on Σ ($U_\alpha \subset \Sigma$ is open). By doing this, we can assume that our Riemann surface is $(\mathbb{C}, i)$, where i is the usual complex structure. Let's give $\mathbb{C}$ the coordinates $z = s + it$. Define $u_\alpha = u \circ \phi_\alpha^{-1}$. In this case we have

$$\begin{aligned}
\bar{\partial}_J u_\alpha &= \frac{1}{2}\left[\left(\frac{\partial u_\alpha}{\partial s} ds + \frac{\partial u_\alpha}{\partial t} dt\right) + J(u_\alpha)\left(\frac{\partial u_\alpha}{\partial t} ds - \frac{\partial u_\alpha}{\partial s} dt\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{\partial u_\alpha}{\partial s} + J(u_\alpha)\frac{\partial u_\alpha}{\partial t}\right) ds + \left(\frac{\partial u_\alpha}{\partial t} - J(u_\alpha)\frac{\partial u_\alpha}{\partial s}\right) dt\right]
\end{aligned}$$

From this, we can see that $\bar{\partial}_J u_\alpha = 0$ if

$$\frac{\partial u_\alpha}{\partial s} + J(u_\alpha)\frac{\partial u_\alpha}{\partial t} = 0 \tag{1}$$

(the dt coefficient is this, multiplied by $J(u_\alpha)$).

Now, if we assume $V = \mathbb{C}^n$ with the usual complex structure i, under the identification $\mathbb{C}^n \cong \mathbb{R}^n \oplus i\mathbb{R}^n$ we get

$$i = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Letting $u_\alpha = f + ig$, equation (1) becomes

$$\left(\frac{\partial f}{\partial s} + i\frac{\partial g}{\partial s}\right) + i\left(\frac{\partial f}{\partial t} + i\frac{\partial g}{\partial t}\right) = \left(\frac{\partial f}{\partial s} - \frac{\partial g}{\partial t}\right) + i\left(\frac{\partial f}{\partial t} + \frac{\partial g}{\partial s}\right) = 0,$$

the familiar Cauchy-Riemann equations (if you like, take $n = 1$).

1.3. Symplectic Manifolds.

Given an even dimensional manifolds V^{2n}, a *symplectic form* on V is a closed, nondegenerate 2-form on V. The nondegeneracy conditions means that, for a vector field X on V, if $\omega(X, Y) = 0$ for all vector fields Y, then $X = 0$. A *symplectic manifold* is a pair (V, ω) where ω is a symplectic form on V.

Example 1. *$\mathbb{C}^n$ with its usual coordinates $z_1, ..., z_n$ is a symplectic manifold with the standard symplectic form*

$$\omega_0 := \sum_{k=1}^{n} dx_k \wedge dy_k,$$

where $z_k = x_k + iy_k$.

Definition 2 (Lagrangian Submanifold)**.** *Given a symplectic manifold (V, ω), a Lagrangian submanifold (or simply, a Lagrangian) in V is a submanifold $L \subset V$ such that $\omega|_{TL} = 0$ (where we consider $TL \subset TV$). Note that a Lagrangian submanifold is necessarily half the dimension of V, that is $\dim L = n$.*

Example 2. *The n-torus $\mathbb{T}^n := \underbrace{S^1 \times \cdots \times S^1}_{n}$ is a Lagrangian submanifold of $(\mathbb{C}^n, \omega_0)$.*

A submanifold $W \subset V$ is called *symplectic* if $\omega|_{TW}$ is again a symplectic form on W.

1.3.1. *Hamiltonian diffeomorphisms.* Let (V, ω) be a symplectic manifold. Given a smooth function $h : V \to \mathbb{R}$, define the *Hamiltonian vector field of f* to be the vector field X_h such that

$$\imath_{X_h}\omega = dh.$$

A *Hamiltonian diffeomorphism* of V is defined to be the time 1 flow, ψ, of a Hamiltonian vector field.

2. Theorems and Applications

2.1. Generalization of the Riemann Mapping Theorem.

Consider again the symplectic manifold $(\mathbb{C}^n, \omega_0)$. Let D denote the unit disc in $\mathbb{C}$. The proof of this result is an application of holomorphic curves, but is quite involved.

Theorem 1 (Gromov '85)**.** *Let $L \subset \mathbb{C}^n$ be a compact Lagrangian submanifold. Then there exists a nonconstant holomorphic disc $u : D \to \mathbb{C}^n$ such that $u(\partial D) \subset L$.*

A corollary of this theorem essentially says that there are always intersections between a Lagrangian submanifold and any Hamiltonian deformation of it (under appropriate assumptions).

Definition 3 (Convex at Infinity)**.** *A noncompact symplectic manifold (V, ω) is called convex at infinity if there exists a pair (f, J), where J is an ω-compatible ($\omega(v, Jv) > 0$ for $v \neq 0$ and $\omega(Jv, Jw) = \omega(v, w)$ for all $x \in V$ and all $v, w \in T_xV$) almost complex structure and $f : V \to [0, \infty)$ is a proper smooth function such that*

$$\omega_f(v, Jv) \geq 0, \qquad \omega_f := -d(df \circ J),$$

for every x outside some compact subset of V and every $v \in T_xV$.

Corollary. *Let (V, ω) be a symplectic manifold without boundary, and assume that (V, ω) is convex at infinity. Let $L \subset V$ be a compact Lagrangian submanifold such that $[\omega]$ vansishes on $\pi_2(V, L)$. Let $\psi : V \to V$ be a Hamiltonian symplectomorphism. Then $\psi(V) \cap V \neq \varnothing$.*

2.2. **The Nonsqueezing Theorem.** Let $B^{2n}(r)$ be the closed ball of radius r and center 0 in $\mathbb{R}^{2n}$. Another application of holomorphic curves is the following

Theorem 2 (Gromov)**.** *If $\iota : B^{2n}(r) \to \mathbb{R}^{2n}$ is a symplectic embedding (the image is a symplectic submanifold of $\mathbb{R}^{2n}$) such that $\iota(B^{2n}(r)) \subset B^2(R) \times \mathbb{R}^{2n-2}$, then $r \leq R$*

and a further generalization of it is

Theorem 3. *Let (V, ω) be a compact symplectic manifold of dimension $2n - 2$ such that $\pi_2(V) = 0$. If there is a symplectic embedding of the ball $(B^{2n}(r), \omega_0)$ into $B^2(R) \times V$, then $r \leq R$.*

References

[1] Mikhail Gromov, *Pseudo holomorphic curves in symplectic manifolds.* Invent. math. 82, 307-347, 1985.
[2] Dusa McDuff and Dietmar Salamon, *J-holomorphic Curves and Symplectic Topology.* Second edition, AMS Colloquium Publications, vol. 52 (2012).