Basic Analysis Introduction Talk
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1. Some Basic Definitions and Facts

1.1. Sequences and Series.

- A sequence of points $\{x_n\}$ in a metric space (X, d) is said to *converge* if there is a point $x \in X$ such that for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n > N$ we have
$$d(x_n, x) < \varepsilon.$$
- A sequence is said to be *bounded* if the sequence, considered as a subset of X is a bounded set.
- Every sequence in a compact metric space has a convergent subsequence.
- A *Cauchy sequence* is a sequence $\{x_n\}$ such that for every $\varepsilon > 0$ there is a number $N \in \mathbb{N}$ such that for all $n, m > N$ we have $d(x_n, x_m) < \varepsilon$.
- In a metric space, every convergent sequence is Cauchy.
- A Cauchy sequence in a compact metric space converges.
- The Cauchy product of two series $\sum a_n$ and $\sum b_n$ is $\sum c_n$ where
$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, n = 0, 1, 2, ...$$

The following deal with sequences of functions.

- Given a sequence of functions $\{f_n\}$ defined on some set E, and suppose that $\{f_n(x)\}$ converges for every $x \in E$, then we may define the limiting function $f(x) = \lim_{n\to\infty} f_n(x)$. This is called *pointwise convergence* of the sequence $\{f_n\}$. Similarly, if the series $\sum f_n(x)$ converges for every $x \in E$, we define the sum of the sequence to be $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Both this limiting function, and the sum are defined on E again.
- A sequence of functions is said to *uniformly converge* to f on E if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for $n > N$ we have
$$|f_n(x) - f(x)| < \varepsilon$$
for all $x \in E$. Uniform convergence of course implies pointwise convergence. We say that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on E if the sequence of partial sums converges uniformly on E.
- The sequence $\{f_n\}$ converges uniformly on E iff for every $\varepsilon > 0$ there is an $N \in N$ such that for $m, n > N$ and $x \in E$ we have
$$|f_n(x) - f_m(x)| < \varepsilon.$$
- Let $\{f_n\}$ be a sequence of functions on E and suppose that $|f_n(x)| \le M_n$ for all $x \in E$. Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

1.2. **Continuity.**

- A function on metric spaces $f : (X, d_x) \to (Y, d_y)$ is *continuous at* $x \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all y with $d_x(x, y) < \delta$ then $d_y(f(x), f(y)) < \varepsilon$. If f is continuous at every $x \in X$, then it is simply called *continuous on* X.
- A function as above is *uniformly continuous* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $d_y(f(x), f(y)) < \varepsilon$ whenever $d_x(x, y) < \delta$.
- The continuous image of a compact set is compact.
- The continuous image of a connected set is connected.
- A continuous function on a compact metric space realizes its minimum and maximum.
- A continuous bijection from a compact metric space to any metric space has a continuous inverse (or, more generally, a continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism).
- A continuous function on a compact metric space is automatically uniformly continuous.
- Mean Value Theorem
- If $\{f_n\}$ uniformly converges to f on a set E in a metric space, then for a limit point $x \in E$ we have
$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

1.3. **Differentiation.**

- Let $\{f_n\}$ be a sequence of differentiable functions on $[a, b]$ such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a, b]$. Then if $\{f_n'\}$ converges uniformly on $[a, b]$ then so does $\{f_n\}$, and moreover, if $f_n \to f$, then
$$f'(x) = \lim_{n\to\infty} f_n'(x).$$
It is important to note that a converse of this is not true, that is $\{f_n\}$ being uniformly convergent says nothing about $\{f_n'\}$. To see this, consider the sequence $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$.
- Let $E \subset \mathbb{R}^n$ be open and suppose $f : E \to \mathbb{R}^m$, and let $x \in E$. Then if there is a linear map $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that
$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0$$
then we say that f is differentiable at x, and we write $f'(x) = A$. As usual, if f is differentiable at every point of E, then we say f is differentiable on E. A is unique. Sometimes $f'(x)$ is called the *total derivative of f at x*. f is said to be continuously differentiable, or C^1, if f' is a continuous mapping of E into $L(\mathbb{R}^n, \mathbb{R}^m)$.

1.4. **Integration.**

- Let α be a monotonically increasing function on $[a, b]$. Suppose $\{f_n\}$ is a sequence of Riemann integrable functions and that $f_n \to f$ uniformly on $[a, b]$. Then f is Riemann integrable and
$$\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha.$$

2. Arzelà-Ascoli Theorem

Let X be a topological space. Denote by $C(X)$ the space of continuous functions from X to $\mathbb{C}$. The *uniform metric* on $C(X)$ is given by $\rho(f, g) = \|f - g\|_u$ where $\|\cdot\|_u$ is the uniform norm given by $\|f\|_u = \sup\{|f(x)| \mid x \in X\}$. A sequence $\{f_n\}$ in $C(X)$ is said to *converge uniformly on compact sets* to a function $f \in C(X)$ if for every compact set $K \subset X$ the sequence $\{f_n|_K\}$ converges uniformly to $f|_K$. We say X is *σ-compact* if it is the countable union of compact sets.

Given a metric space X, we say that a subset of X is *totally bounded* if for every $\varepsilon > 0$, X can be covered by finitely many metric balls of radius ε.

Let X be a topological space and let $\mathcal{F} \subset C(X)$. We say that $\mathscr{F}$ is *equicontinuous at* $x \in X$ if for every $\varepsilon > 0$ there is a neighborhood U of x such that $|f(y) - f(x)| < \varepsilon$ for all $y \in U$ and $f \in \mathcal{F}$. $\mathcal{F}$ is simply called *equicontinuous* if it is equicontinuous at all $x \in X$. Furthermore, we say that $\mathcal{F}$ is *pointwise bounded* if $\{f(x) \mid f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$ for all $x \in X$.

We will cover two versions of the Arzelà-Ascoli theorem:

Theorem 1 (Arzelà-Ascoli I)**.** *Let X be a compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $C(X)$, then $\mathcal{F}$ is totally bounded in the uniform metric, and the closure of $\mathcal{F}$ in $C(X)$ is compact.*

Theorem 2 (Arzelà-Ascoli II)**.** *Let X be a σ-compact locally compact, Hausdorff space. If $\{f_n\}$ is an equicontinuous, pointwise bounded sequence in $C(X)$, then there exists an $f \in C(X)$ and a subsequence of $\{f_n\}$ that converges to f uniformly on compact sets.*

Example (Folland, Ch. 4, Ex. 64)**.** *Let (X, ρ) be a metric space. A function $f \in C(X)$ is called* Hölder continuous of exponent α *($\alpha > 0$) if the quantity*

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\alpha}$$

is finite. If X is compact, $\{f \in C(X) \mid \|f\|_u \leq 1$ and $N_\alpha(f) \leq 1\}$ is compact in $C(X)$.

Proof. Let $\mathcal{F}$ be the set we are trying to show is compact in $C(X)$. Note that a metric space is Hausdorff. The COA for this problem is to show that $\mathcal{F}$ is closed, equicontinuous, and pointwise bounded.

First, since $\|f\|_u \leq 1$ for all $f \in \mathcal{F}$, we know that for all $x \in X$, $\{f(x) \mid f \in \mathcal{F}\}$ is bounded in $\mathbb{C}$. Hence $\mathcal{F}$ is pointwise bounded.

Second, note that, since for all $f \in \mathcal{F}$ we have $N_\alpha(f) \leq 1$, by definition of $N_\alpha(f)$ we have $|f(x) - f(y)| \leq \rho(x, y)^\alpha$. Fix $x \in X$ and let $\varepsilon > 0$. Let $U = B_{\varepsilon^{\frac{1}{\alpha}}}(x)$, then for all $y \in U$ we have

$$|f(x) - f(y)| \leq \left(\varepsilon^{\frac{1}{\alpha}}\right)^\alpha = \varepsilon.$$

Therefore $\mathcal{F}$ is equicontinuous at $x \in X$. This same argument holds for all $x \in X$, and therefore $\mathcal{F}$ is equicontinuous.

Therefore, by Arzelà-Ascoli I, $\overline{\mathcal{F}}$ is compact in $C(X)$.

Now, let $\{f_n\}$ be a sequence in $\mathcal{F}$ converging to a function $f \in C(X)$. Since

$$\begin{aligned}
\|f\|_u &= \|\lim_{n\to\infty} f_n\|_u \\
&= \sup_{x\in X}\left|\lim_{n\to\infty} f_n(x)\right| \\
&= \sup_{x\in X}\lim_{n\to\infty} |f_n(x)| \\
&= \lim_{n\to\infty}\sup_{x\in X} |f_n(x)| \\
&= \lim_{n\to\infty} \|f_n\|_u \\
&\leq \lim_{n\to\infty} 1 = 1
\end{aligned}$$

and

$$\begin{aligned}
N_\alpha(f) &= \sup_{x\neq y}\frac{|f(x)-f(y)|}{\rho(x,y)^\alpha} \\
&= \sup_{x\neq y}\frac{|\lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(y)|}{\rho(x,y)^\alpha} \\
&= \sup_{x\neq y}\frac{|\lim_{n\to\infty} [f_n(x) - f_n(y)]|}{\rho(x,y)^\alpha} \\
&= \sup_{x\neq y}\lim_{n\to\infty}\frac{|f_n(x)-f_n(y)|}{\rho(x,y)^\alpha} \\
&= \lim_{n\to\infty}\sup_{x\neq y}\frac{|f_n(x)-f_n(y)|}{\rho(x,y)^\alpha} \\
&= \lim_{n\to\infty} N_\alpha(f_n) \\
&\leq \lim_{n\to\infty} 1 = 1
\end{aligned}$$

we thus have that $f \in \mathcal{F}$, and hence $\mathcal{F}$ is closed, and by the above, compact in $C(X)$. □

3. Stone-Weierstrauβ Theorem

Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a subset of $C(X,\mathbb{R})$ or $C(X)$. We say that $\mathcal{A}$ *separates points* if for all $x, y \in X$ there is a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. If $\mathcal{A}$ is a real (resp. complex) vector subspace of $C(X,\mathbb{R})$ (resp. $C(X)$) such that when $f, g \in \mathcal{A}$, $fg \in \mathcal{A}$, we call $\mathcal{A}$ an *algebra*.

Theorem 3 (Stone-Weierstrauβ)**.** *Let X be a compact Hausdorff space. If $\mathcal{A}$ is a closed subalgebra of $C(X,\mathbb{R})$ that separates points, then either $\mathcal{A} = C(X,\mathbb{R})$ or $\mathcal{A} = \{f \in C(X,\mathbb{R}) \mid f(x_0) = 0\}$ for some $x_0 \in X$. The first alternative holds iff $\mathcal{A}$ contains the constant functions.*

Theorem 4 (Complex Stone-Weierstrauβ)**.** *Let X be a compact Hausdorff space. If $\mathcal{A}$ is a closed complex subalgebra of $C(X)$ that separates points and is closed under complex conjugation, then either $\mathcal{A} = C(X)$ or $\mathcal{A} = \{f \in C(X) \mid f(x_0) = 0\}$ for some $x_0 \in X$.*

Example (Folland, Ch. 4, Ex. 68)**.** *Let X and Y be compact Hausdorff spaces. The algebra generated by functions of the form $f(x,y) = g(x)h(y)$, where $g \in C(X)$ and $h \in C(Y)$, is dense in $C(X \times Y)$.*

Proof. Let $\mathcal{A}$ be the algebra in the statement. Clearly $\mathcal{A}$ is a complex subalgebra of $C(X \times Y)$ and it is also clearly closed under complex conjugation. Of course, what we really need to show is that the closure of $\mathcal{A}$, $\overline{\mathcal{A}}$, is equal to $C(X)$. Since complex algebra operations are continuous, and conjugation is continuous, we also have that $\overline{\mathcal{A}}$ is a complex subalgebra of $C(X \times Y)$. Now clearly $\overline{\mathcal{A}}$ contains the constant functions since we may take g and h to be any constant function on X and Y respectively. Furthermore given two distinct points $(x_1, y_1), (x_2, y_2) \in X \times Y$ (WLOG assume that $x_1 \neq x_2$), choosing a nonzero constant function for $h(y)$ and a function $g(x)$ which takes different values on x_1 and x_2 (which is possible since $C(X)$ itself separates points, we have a function $f(x, y) = g(x)h(y)$ having different values on (x_1, y_1) and (x_2, y_2), hence $\overline{\mathcal{A}}$ separates points. Applying the complex version of Stone-Weierstrauβ to $\overline{\mathcal{A}}$ we now have that $\overline{\mathcal{A}} = C(X \times Y)$, meaning that $\mathcal{A}$ is dense in $C(X \times Y)$. □

4. The Inverse and Implicit Function Theorems

Let us recall that a contraction mapping from a metric space (X, ρ) to itself is a map $\phi : X \to X$ such that there is a number $c < 1$ such that $\rho(\phi(x), \phi(y)) \leq c\rho(x, y)$ for all $x, y \in X$. A simple, yet powerful result is the following:

Theorem 5 (The Contraction Principle)**.** *If X is a complete metric space, and if ϕ is a contraction of X into itself, then there exists a unique fixed point of ϕ.*

Proof. Let $\phi : X \to X$ be our contraction mapping, and let c be the appropriate constant. Let's begin by proving the uniqueness of the fixed point. Say there are two distinct points, $x, y \in X$, such that $\phi(x) = x$ and $\phi(y) = y$. Then $\rho(x, y) = \rho(\phi(x), \phi(y)) \leq c\rho(x, y) < \rho(x, y)$, a contradiction. Thus the fixed point, if it exists, is unique.

Now, for the existence. Choose a point $x_0 \in X$, and define $x_{n+1} := \phi(x_n)$. Note that $\rho(x_{n+1}, x_n) = c^n\rho(x_1, x_0)$. Now, for arbitrary $n < m$ we have

$$\begin{aligned}
\rho(x_m, x_n) &\overset{\Delta-ineq.}{\leq} \sum_{i=n+1}^{m} \rho(x_i, x_{i-1}) \\
&\leq \sum_{i=n+1}^{m} c^{i-1}\rho(x_1, x_0) \\
&= (c^n + c^{n+1} + \cdots + c^{m-1})\rho(x_1, x_0) \\
&\leq \frac{c^n}{1-c}\rho(x_1, x_0)
\end{aligned}$$

so, since $c < 1$, this shows that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, this sequence converges, say to x. Now

$$\phi(x) = \phi\left(\lim_{n\to\infty} x_n\right) \overset{\phi \text{ cont.}}{=} \lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} x_{n+1} = x,$$

thus showing that ϕ has a fixed point.

□

This next theorem is used for finding "local inverses" of functions. Simply put, if the derivative matrix of a function is invertible at a point, then the function is invertible in a neighborhood of that point

Theorem 6 (The Inverse Function Theorem)**.** *Let $E \subset \mathbb{R}^n$ be an open set and suppose that $f \in C^1(E, \mathbb{R}^n)$, $f'(a)$ is invertible for some $a \in E$, and $f(a) = b$. Then*

(a) there exist open sets U and V in $\mathbb{R}^n$ such that $a \in U$, $b \in V$, f is injective on U, and $f(U) = V$;

(b) if g is the inverse of f (which exists by (a)), defined in V by

$$g(f(x)) = x, \quad x \in U,$$

then $g \in C^1(V)$.

A nice consequence of this theorem (really just part (a) of it) is the following:

Corollary. *Let f be as in the theorem, and suppose that f' is invertible for all $x \in E$, then $f(W)$ is open in $\mathbb{R}^n$ for all open $W \subset E$.*

Proof. Let W an open subset of E (and hence it is an open subset of $\mathbb{R}^n$), restrict f to W, and for each $x \in W$ let U_x and V_x be the neighborhoods of x and $f(x)$, respectively, given by the inverse function theorem. Then $\bigcup_{x \in W} U_x = W$, and we have

$$f(W) = f\left(\bigcup_{x \in W} U_x\right) = \bigcup_{x \in W} f(U_x) = \bigcup_{x \in W} V_x$$

showing thus that $f(W)$ is an open set in $\mathbb{R}^n$. ☐

Here is a simple, but illustrating example of a possible pitfall one could fall into with the inverse function theorem.

Example (Adapted from Rudin, Ch. 9, Ex. 17)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (e^x \cos y, e^x \sin y)$. Show that f' is everywhere invertible, but that f does not have a global inverse.*

Proof. It is clear that f is a C^1 function, so let's begin by computing $f'(x, y)$:

$$f'(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

Then since we have

$$\det f'(x, y) = e^{2x} \cos^2 y + e^{2x} \sin^2 y = e^{2x}$$

which is never zero, we see that $f'(x, y)$ is everywhere invertible. However, f cannot have a global inverse since it is clearly not injective (consider the two points $(0, 0)$ and $(0, 2\pi)$, for example). What this shows is that f has a local inverse at every point of $\mathbb{R}^2$, but no global inverse exists. ☐

Notation 1. *For the implicit function theorem, we will refer to points in $\mathbb{R}^{n+m}$ as pairs $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. So, for $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and $\mathbf{h} \in \mathbb{R}^m$ and $\mathbf{k} \in \mathbb{R}^n$, we define $A_x\mathbf{h} := A(\mathbf{h}, \mathbf{0})$ and $A_y\mathbf{k} := A(\mathbf{0}, \mathbf{k})$.*

Theorem 7 (The Implicit Function Theorem)**.** *Let $E \subset \mathbb{R}^{n+m}$ be an open set and let $f \in C^1(E, \mathbb{R}^n)$ such that $f(a, b) = 0$ for some $(a, b) \in E$. Let $A = f'(a, b)$ and assume that A_x is invertible. Then there are open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$, with $(a, b) \in U$ such that the following property holds: For every $y \in W$ there is a unique x such that $(x, y) \in U$ and $f(x, y) = 0$. Letting $x = g(y)$, then $g \in C^1(W, \mathbb{R}^n)$, $g(b) = a$, $f(g(y), y) = 0$ for $y \in W$, and $g'(b) = -(A_x)^{-1}A_y$.*

5. Fourier Series

A trigonometric polynomial is one of the form

$$f(x) = a_0 + \sum_{n=-N}^{N} (a_n \sin nx + b_n \cos nx)$$

where $x \in R$ and $a_n, b_n \in \mathbb{C}$. More compactly we can write this as

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx}. \tag{1}$$

Clearly these polynomials have period 2π. Note now that

$$\int_{-\pi}^{\pi} e^{inx} = \begin{cases} 2\pi, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Multiplying (1) by e^{-imx} then integrating from $-\pi$ to π we see that, by the above,

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx,$$

for $|m| \leq N$ and $c_m = 0$ otherwise. From this, we determine that (1) is real iff $c_{-n} = \overline{c_n}$. Accordingly, we now define a trigonometric series by

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{2}$$

For a Riemann integrable function f with period 2π, we call the c_n it's *Fourier coefficients*, and the associated series its *Fourier series*. To tie a function with its Fourier series, we shall write $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$.

I don't wish to delve into Fourier analysis here, but I do wish to point out a rather interesting consequence:

Theorem 8 (Parseval's Theorem)**.** *If $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ and $g(x) \sim \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}$, then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx = \sum_{n=-\infty}^{\infty} c_n \overline{\gamma_n},$$

and, in particular,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

6. Miscellany

Just a quick comment on something incredibly useful. Recall that the support of a real valued function is the closure of the set of points where it is nonzero, i.e., for $f : X \to \mathbb{R}$, $\text{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$.

Theorem 9 (Partition of Unity)**.** *Suppose that $K \subset \mathbb{R}^n$ is compact, and let $\{V_\alpha\}$ be an open cover of K. Then there are functions $\psi_1, ..., \psi_s \in C(\mathbb{R}^n)$ such that*

(1) $0 \leq \psi_i \leq 1$, for $1 \leq i \leq s$
(2) each ψ_i has its support in some V_α
(3) $\psi_1(x) + \cdots + \psi_s(x) = 1$ for all $x \in K$.

Because of (3) $\{\psi_i\}$ is called a partition of unity, and (2) is said as $\{\psi_i\}$ is subordinate to $\{V_\alpha\}$.

Recall that the Jacobian of a function $f : E \to R^n$ where E is open in R^n is given by $J_f(x) = det f'(x)$.

Theorem 10 (Change of Variables). *Suppose T is an injective C^1 mapping of an open set E in $\mathbb{R}^k$ into $\mathbb{R}^k$ such that $J_T(x) \neq 0$ for all $x \in E$. If f is a continuous function on $\mathbb{R}^k$ with compact support in $T(E)$, then*

$$\int_{\mathbb{R}^k} f(y)dy = \int_{\mathbb{R}^k} f(T(x))|J_T(x)|dx.$$

7. Stokes' Theorem

Theorem 11 (Stokes' Theorem). *If Ψ is a C^2 smooth k-chain in an open set $V \subset \mathbb{R}^m$ and if ω is a C^1 $(k-1)-form$ in V, then*

$$\int_{\Psi} d\omega = \int_{\partial\Psi} \omega.$$

Corollary (Green's Theorem). *Suppose E is an open set in $\mathbb{R}^2$, $\alpha \in C^1(E)$, $\beta \in C^1(E)$, and Ω a closed subset of E with positively oriented boundary $\partial\Omega$. Then*

$$\int_{\partial\Omega} \alpha dx + \beta dy = \int_{\Omega} (\beta_x - \alpha_y) dA.$$

References

[1] Gerald B. Folland, *Real Analysis.*
[2] Walter Rudin, *Principles of Mathematical Analysis.*