Basic Topology Introduction Talk
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1. Point-Set Topology

1.1. Definitions, Constructions, and Examples.

Definition 1 (Topology)**.** Given a set X, a *topology* on X is a collection of subsets, $\mathcal{T}$, of X such that:

- $\varnothing, X \in \mathcal{T}$
- the union of an arbitrary collection of elements of $\mathcal{T}$ is again an element of $\mathcal{T}$
- the intersection of a finite collection of elements of $\mathcal{T}$ is again an element of $\mathcal{T}$

The pair $(X, \mathcal{T})$ is called a *topological space*, and elements of $\mathcal{T}$ are called *open sets*. The complement of an open set is called a *closed set*, and vice versa. Note: It is common to suppress the $\mathcal{T}$ if the topology on X is understood.

Some examples of topological spaces:

(1) The *indiscreet topology* or *trivial topology* on a set X is given by $\mathcal{T} = \{\varnothing, X\}$.
(2) The *discreet topology* on a set X is given by $\mathcal{T} = \mathcal{P}(X)$ (the power set of X).
(3) On the set $X = \{a, b, c\}$, the collection $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, X\}$ is a topology.

Since this definition can often be unwieldy, it is usually convenient to describe a topology in terms of a *basis*.

Definition 2 (Basis for a topology)**.** Let X be a set and let $\mathcal{B}$ be a collection of subsets of X such that

- for every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$
- given $B_1, B_2 \in \mathcal{B}$, there is a $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$

Then the collection $\mathcal{T} = \{\cup \mathcal{U} \mid \mathcal{U} \subset \mathcal{B}\}$ is a topology on X, and we say $\mathcal{B}$ is a *basis* for the topology $\mathcal{T}$.

Some more examples of topological spaces:

(1) $\mathbb{R}^n$ with its usual topology given by the basis $\mathcal{B} = \{B_\varepsilon(x) \mid x \in \mathbb{R}^n, \varepsilon > 0\}$ consisting of open balls. (Recall $B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid |x - y| < \varepsilon\}$.)
(2) Any metric space (X, d) is a topological space whose topology is given by the basis consisting of all open balls in the metric space.

Given a new construction, there should be two natural questions that you ask about them:

- what are "morphisms" (structure preserving maps) between them?
- how do you make new ones out of old ones?

Let's answer those questions in that order.

1.1.1. *Maps between topological spaces.* A morphism in the category of topological spaces is a continuous map:

Definition 3 (Continuous function)**.** Given two topological spaces X and Y, and a function $f : X \to Y$ between the sets X and Y, we say that f is *continuous* if for all open sets V in Y, the preimage $f^{-1}(V)$ is an open subset of X.

Now that we have a way to map between topological spaces, what about the notion of isomorphism?

Definition 4 (Homeomorphism)**.** Let $f : X \to Y$ be a continuous bijection. If $f^{-1} : Y \to X$ is also a continuous map, then we say that f is a *homeomorphism.* If X and Y are homeomorphic, we often write $X \approx Y$.

1.1.2. *New spaces out of old.* One way we can make a new space is to combine two old ones:

Definition 5 (Product topology)**.** Given two topological spaces X and Y, we define a topology on the product $X \times Y$ via the basis

$$\mathcal{B} = \{U \times V \mid U \underset{open}{\subset} X, V \underset{open}{\subset} Y\}.$$

This basis gives the *product topology* on $X \times Y$.

Common examples:

(1) $\mathbb{R}^m \times \mathbb{R}^n (\approx \mathbb{R}^{m+n})$ with their usual topologies
(2) The torus $\mathbb{T}^2 := S^1 \times S^1$ (we'll describe the topology on S^1 in a moment)

Another way we can make a new space is to take subspaces:

Definition 6 (Subspace topology)**.** Let X be a topological space and let $S \subset X$. Then the *subspace topology* on S is given by open sets of the form $U \cap S$ for $U \underset{open}{\subset} X$.

More examples:

(1) Any of the matrix groups viewed as subsets of the proper $\mathbb{R}^{2n}$ or $\mathbb{C}^{2n}$, e.g., $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O(n)$, $SO(n)$ as subsets of $\mathbb{R}^{2n}$; or $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $U(n)$, $SU(n)$ as subsets of $\mathbb{C}^{2n}$.
(2) Additionally, the additive group of $m \times n$ matrices (say, over $\mathbb{R}$) regarded as a subset of $\mathbb{R}^{mn}$
(3) The n-sphere: $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$
(4) The n-disc: $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$
(5) The Stiefel manifold, $V_k(\mathbb{R}^n) = \{(v_1, ..., v_k) \mid v_i \in \mathbb{R}^n; \langle v_i, v_j \rangle = 0, i \neq j\}$, can be viewed as a subset of $\mathbb{R}^{nk}$

One more way is to take quotients of a topological space:

Definition 7 (Quotient topology)**.** Let X be a topological space and let $\sim$ be an equivalence relation on X. Denote by $X/\sim$ the set of equivalence classes and let $p : X \to X/\sim$, $x \mapsto [x]$ be the natural projection map. The *quotient topology* on $X/\sim$ is given by

$$\mathcal{T} = \{U \subset X/\sim \ \mid p^{-1}(U) \underset{open}{\subset} X\}.$$

Fact: when $X/\sim$ has the quotient topology, the projection map $p : X \to X/\sim$ is a continuous map (by construction).

Examples:

(1) Let X be a topological space and let $A \subset X$. Define an equivalence relation $\sim$ on X by $x \sim y$ if $x = y$ or $x, y \in A$. Then the quotient space $X/\sim$, usually denoted X/A, is the space formed by collapsing all of A to a single point.
(2) $D^n/S^{n-1} \approx S^n$
(3) Doing various edge identifications on a square will yield, for example: a torus, a Klein bottle, a 2-sphere, or a real projective plane.
(4) The real projective space $\mathbb{RP}^n := S^n/(v \sim \pm v)$ (this is the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$). We also have $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/(x \sim cx)$, $c \in \mathbb{R} \setminus \{0\}$.
(5) The complex projective space $\mathbb{CP}^n := S^{2n+1}/(v \sim \lambda v)$, $\lambda \in \S^1$ (this is the set of 1-dimensional complex subspaces of $\mathbb{C}^{n+1}$). We also have $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(x \sim cx)$, $c \in \mathbb{C} \setminus \{0\}$.

(6) The Grassmann manifold $G_k(\mathbb{R}^{n+k})$, the space of k-dimensional subspaces in $\mathbb{R}^{n+k}$, can be formed from the Stiefel manifold $V_k(\mathbb{R}^{n+k})$ by taking the quotient by the equivalence relation $U \sim V$ if $span(U) = span(V)$.

1.1.3. *Properties.*

Definition 8 (Second countable)**.** A topological space is said to be *second countable* if it has a basis for its topology consisting of a countable number of subsets.

Definition 9 (Hausdorff)**.** A topological space X is called *Hausdorff*, or T_2, if for all $x, y \in X$, there are disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.

Definition 10 (Open cover)**.** Given a topological space X, a collection $\mathcal{U}$ of open sets in X such that the union of all the sets in $\mathcal{U}$ is all of X is called an *open cover* of X.

Definition 11 (Compact)**.** A topological space X is called *compact* if given any open cover $\mathcal{U}$ of X, there is a finite subcollection $\mathcal{U}' \subset \mathcal{U}$ such that $\mathcal{U}'$ still covers X.

Definition 12 (Connected)**.** A topological space X is called *connected* if it cannot be written as a union of a pair of disjoint open subsets.

Definition 13 (Path connected)**.** A topological space X is called *path connected* if given any two elements $x, y \in X$, there is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Such a continuous function is called a *path* from x to y.

Note that a path connected space is always connected, but it is not true that a connected space is path connected. A counter example is the following, known as the *topologist's sine curve*:
Let $S = (\{0\} \times [-1, 1]) \cup \left\{\left(x, \sin \frac{1}{x}\right) \mid 0 < x \leq 1\right\} \subset \mathbb{R}^2$. Then S is connected, but not path connected.

1.2. Some important facts.

Lemma 14. *Let $\sim$ be an equivalence relation on a topological space X and let $X/\sim$ be the quotient space. Let $p : X \to X/\sim$ be the projection map. A map $f : X/\sim \to Y$ to a topological space Y is continuous if and only if the lift $f \circ p : X \to Y$ is continuous.*

Lemma 15. *Let $f : X \to Y$ be continuous. If $C \subset X$ is (path) connected, then so is $f(C)$; if C is compact, then so is $f(C)$.*

Lemma 16.

- *Let X be a compact space and let K be a closed subspace. Then K is compact.*
- *Let X be a Hausdorff space and let K be a compact subspace. Then K is closed.*

Corollary 17. *A continuous bijection $f : X \to Y$ from a compact space X to a Hausdorff space Y is a homeomorphism.*

A useful characterization of compact spaces inside of $\mathbb{R}^n$ is the following:

Theorem 18 (Heine-Borel Theorem)**.** *A subspace $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded. (A subspace being bounded means there is an $\varepsilon > 0$ such that $X \subset B_\varepsilon(0)$.)*

Lemma 19. *A finite product of (path) connected spaces is (path) connected.*

Theorem 20 (Tychonoff's theorem)**.** *An arbitrary product of compact spaces is compact.*

1.3. **Topological groups.** One last construction for this section: topological groups.

Let $(G,\cdot)$ be a group. Define the maps $\mu : G\times G \to G$ and $\iota : G \to G$ to be the multiplication and inversion maps on G, i.e.,

$$\mu(g,h) = g\cdot h, \quad \iota(g) = g^{-1}.$$

Definition 21 (Topological group)**.** Let $(G,\cdot)$ be a group. If there is a topology $\mathcal{T}$ on G such that μ and ι are continuous maps, then $(G,\mathcal{T})$ is called a *topological group*.

Examples:

(1) $(S^1,\cdot)$, $(\mathbb{Z},+)$, $(\mathbb{R},+)$, $(\mathbb{C}^*,\cdot)$, etc.
(2) $GL_n(\mathbb{R})$, $SL_n(\mathbb{C})$, $O(n)$, $U(n)$, $SO(n)$, etc.

2. Homotopy

It is often difficult in topology to prove things up to homeomorphism. Often, we only prove stuff up to homotopy. In fact, much of algebraic topology classifies topological spaces up to homotopy. From here onward, we denote the unit interval by I.

Definition 22 (Homotopy)**.** Let X and Y be topological spaces. A *homotopy* between continuous maps $f : X \to Y$ and $g : X \to Y$ is a continuous map $H : X\times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$. We often write $f \simeq g$ for "f is homotopic to g". If the restriction of H to a subset $A \subset X$ is independent of t, then we say that H is a homotopy *relative to* A between f and g. The notation for this is often $f \simeq g$ (rel A).

Definition 23 (Deformation retraction)**.** Let X be a topological space and A a subspace. Let $r : X \to X$ be a continuous map such that $r(X) = A$ and $r|_A = \mathrm{id}_A$ (this kind of map is called a *retraction*). A homotopy $H : X \times I \to X$ such that $H(x,0) = \mathrm{id}_X$ and $H(x,1) = r(x)$ is called a *deformation retraction* of X onto the subspace A.

Examples of deformation retractions:

(1) $\mathbb{R}^n$ deformation retracts onto any one point subspace
(2) $\mathbb{R}^{n+1} \setminus \{0\}$ deformation retracts onto the n-sphere S^n

Definition 24 (Homotopy equivalence)**.** Let X and Y be topological spaces. A map $f : X \to Y$ is called a *homotopy equivalence* if there is a map $g : Y \to X$ such that $f\circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. If there exists a homotopy equivalence between two spaces, they are said to be *homotopy equivalent* or to have the same *homotopy type*.

One particular example to point out here is that if a space has a deformation retraction onto one of its subspaces, then the whole space and the subspace have the same homotopy type. Let $r : X \to A$ be the retraction and let $\iota : A \hookrightarrow X$ be the inclusion map. Then $r\circ\iota = \mathrm{id}_A$ and $\iota\circ r \simeq \mathrm{id}_X$ (via the deformation retraction since $\iota\circ r = r$, but just expands the codomain to all of X). This is often useful for computing homotopy and (co)homology groups.

Definition 25 (Contractible)**.** If a space has the homotopy type of a point, it is called *contractible*. An equivalent way to say this is that the identity map of the space be *nullhomotopic*, that is, homotopic to a constant map.

2.1. **The Fundamental Group π_1.** Here we will focus specifically on homotopies between paths with fixed endpoints in a space X. This is sometimes called a *path homotopy.* These homotopies will take the form $H : I \times I \to X$ (since paths in X are maps of the form $\gamma : I \to X$) where $H(s, 0)$ and $H(s, 1)$ are constant.

Property 26. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.*

Via this proposition, we may define the *homotopy class*, $[\gamma]$ of a path γ. Next, we define a product on paths.

Definition 27 (Concatenation)**.** Let $\gamma, \delta : I \to X$ be paths in a space X such that $\gamma(1) = \delta(0)$. Define their *concatenation* (or *product*) to be the path

$$\gamma \cdot \delta(s) = \begin{cases} \gamma(2s), & 0 \leq s \leq \frac{1}{2} \\ \delta(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Now, let's restrict the class of paths we look at to *loops.* A *loop* in a space X is a path whose endpoints are the same, and we refer to this point as the *basepoint.* Denote by $\pi_1(X, x_0)$ the set of homotopy classes of loops in X with basepoint x_0.

Property 28. *The set $\pi_1(X, x_0)$ is a group under the product $[\gamma][\delta] = [\gamma \cdot \delta]$.*

$\pi_1(X, x_0)$ is called the *fundamental group* of the space X at basepoint x_0. Note that it be shown that the fundamental group of a path connected space is independent of baseboint. In this case, we usually just write $\pi_1(X)$.

Property 29. *Taking fundamental groups is a functor*

$$\pi_1 : Top_* \to Grp$$

Definition 30 (Simply-connected)**.** We say that a path connected space X is *simply-connected* if $\pi_1(X) = \{e\}$.

Examples:

(1) Some examples of simply connected spaces are $\mathbb{R}^n$, S^n for $n \geq 2$, D^n
(2) $\pi_1(S^1) \cong \mathbb{Z}$
(3) $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$

Given a map $f : (X, x_0) \to (Y, y_0)$ (meaning $f(x_0) = y_0$), we get an induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ which is given by composition with f, i.e., for an element $[\gamma] \in \pi_1(X, x_0)$, $f_*[\gamma] = [f \circ \gamma]$.

2.2. **Seifert-Van Kampen Theorem.** To state the theorem, we need to first make some algebraic definitions:

Definition 31 (Free product)**.** Given two groups G and H, we define the *free product*, $G * H$, as the group of reduced words in G and H, with multiplication given by concatenation followed by reduction.

- Every element of $G * H$ is of the form $g_1 h_1 g_2 h_2 \cdots$ where $g_i \in G$ and $h_j \in H$, that is, the elements alternate between those of G and H.
- Reduced word in G and H are obtained as follows:
  (1) take a word using the elements of $G \cup H$
  (2) remove any instances of the identity (in G or H) from the word
  (3) replace any pair $g_1 g_2$ by its product in G, and likewise for pairs $h_1 h_2$
  (4) repeat steps (2) and (3) as necessary
- Another way to see the free product is as follows:

(1) Let $G = \langle R_G | S_G \rangle$ and $H = \langle R_H | S_H \rangle$ be presentations of G and H.
(2) $G * H = \langle R_G \cup R_H | S_G \cup S_H \rangle$.

Definition 32 (Amalgamated free product)**.** Let G, H, and K be groups, and let $\phi : K \to G$ and $\psi : K \to H$ be group homomorphisms. We define the *amalgamated free product*, $G *_K H$ to be the group $G * H$ with the added relations $\phi(k)\psi(k)^{-1} = e$. That is, it is formed by the following process:

(1) First take the free product $G * H$
(2) Let N be the smallest normal subgroup of $G * H$ generated by elements of the form $\phi(k)\psi(k)^{-1}$
(3) Take the quotient: $G *_K H := (G * H)/N$

Note that, in the category of groups, the amalgamated free product, $G *_K H$ is the pushout of the diagram (in the notation from the definition)

$$\begin{array}{ccc} K & \xrightarrow{\psi} & H \\ {\scriptstyle\phi}\downarrow & & \\ G & & \end{array}$$

Theorem 33 (Seifert-Van Kampen)**.** *Let X be a topological space which is the union of two path connected open sets U and V both containing the base point x_0 and such that $U \cap V$ is path connected. Then there is an isomorphism*

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0).$$

Proof. We will do this proof via category theory. It's not the most intuitive or enlightening proof, but it is a nice application of the theory. First note that we have the following set up:

$$\begin{array}{ccc} U \cap V & \xrightarrow{j} & V \\ \downarrow{\scriptstyle i} & & \\ U & & \end{array}$$

Applying the functor π_1 to this gives us

$$\begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{j_*} & \pi_1(V, x_0) \\ {\scriptstyle i_*}\downarrow & & \\ \pi_1(U, x_0) & & \end{array}$$

We now take pushouts in both diagrams, giving

$$\begin{array}{ccc} U \cap V & \xrightarrow{j} & V \\ {\scriptstyle i}\downarrow & & \downarrow \\ U & \longrightarrow & U \cup V = X \end{array} \qquad \begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{j_*} & \pi_1(V, x_0) \\ {\scriptstyle i_*}\downarrow & & \downarrow \\ \pi_1(U, x_0) & \longrightarrow & \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \end{array}$$

Applying π_1 to the pushed-out diagram in Top_* gives us

$$\begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{j_*} & \pi_1(V, x_0) \\ {\scriptstyle i_*}\downarrow & & \downarrow \\ \pi_1(U, x_0) & \longrightarrow & \pi_1(X, x_0) \end{array}$$

and thus by universality of pushouts, we get the desired isomorphism. □

Corollary 34. *In the set up of the Seifert-Van Kampen theorem, if the intersection $U \cap V$ is simply connected, then the isomorphism is much simpler:*

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0).$$

References

[1] Allen Hatcher, *Algebraic Topology.*
[2] James R. Munkres, *Topology.*
[3] Stephan Stolz, *Notes on point set topology.*