Ringoids
By Edward Burkard

This paper is intended to be more of a "survey" of the theory of ringoids. This paper was possible in no small part due to the tremendous amount of help from Dr. John Baez. In the second part of the paper I am following quite closely the article by S.K. Sehgal titled "Ringoids with minimum condition" with very few exceptions (these exceptions are just examples really and a few details on some proofs). The main goal of the article is to establish the *Wedderburn Theorem for Simple Ringoids*.

Question: Consider the category of abelian groups. We want to define a "ring-like" structure, that is, "ring-like" in the sense that a groupoid is "group-like". What would be the morphisms in this case? How could we define a partial addition and a partial multiplication on this category.

Answer: Let the morphism between any two elements G_i and G_j of the category be the abelian group of homomorphisms between the two groups. Define the addition on this category as addition of homomorphisms. This gives us a partially defined addition since two morphisms can only be added if they are in the same abelian group of homomorphisms. Define the multiplication on this category as the function composition of two homomorphisms. Then this gives a partially defined multiplication in the sense that two homomorphisms can only be composed if the range space of the first homomorphism is a subset of the domain of the second (notice that this is similar to the criterion for two elements in a groupoid to be composable). Recall that in a ring, not every element is invertible (in general). Now notice that every morphism here is not necessarily invertible as not all homomorphisms are bijections so inverses are not necessarily defined. The other ring properties hold as well, that is, when they make sense (i.e. distributivity and associativity).

So we can roughly think of a ringoid as a "collection" of elements with operations of multiplication and addition defined on certain ordered pairs and having the ring properties, whenever the operations are defined.

Thinking of it in a categorical sense:

Definition 1 *An* enriched category *is a category whose hom-sets are replaced by objects from another category, in a well-behaved manner.*

Definition 2 *A* ringoid R *is a category enriched over the category of abelian groups.*

Example 3 *An interesting ringoid is the one where the units are the natural numbers and the morphisms are additive abelian groups. So the morphism between the units* m *and* n *is the additive abelian group of* mxn *matrices.*

Remark 4 *If K is a preadditive category, it maps from a ringoid. A map between ringoids is a functor (since ringoids are categories) that is an abelian group homomorphism on each hom-set. In fact a preadditive category is a ringoid. The definition of a* preadditive category *is: a category enriched over the monoidal category of abelian groups.*

Some examples:

Exercise 5 *Given two ringoids R and S, there is a category* $\hom(R, S)$ *where the objects are maps from R to S, and the morphisms are natural transformations between such maps. Then* $\hom(R, S)$ *is again a ringoid.*

Example 6 *Consider the first example of a ringoid given above. It was the category* **Ab**. *More precisely,* **Ab** *is a closed monoidal category. (A category is* closed *if it is enriched over itself.)*

Example 7 *The category of (left) modules over a ring R (in particular the category of vector spaces over a field F).*

Now to start following the article:

PRELIMINARIES

Axiom 8 *A collection, R, of elements is said to be a* ringoid *if the operations of addition and multiplication are defined for certain pairs of elements and satisfy the following axioms:*

1) $R = \bigsqcup_{i \epsilon I} G_i$, where the G_i are non-trivial, additive abelian groups and I is an index set.

2) Addition is only defined between elements that belong to the same G_i and (the rules for multiplication are similar to that of addition in a groupoid) multiplication, which can occur between any two G_i, is defined whenever it makes sense. (Admittedly this is a little vague, so I will give an example of this...)

3) For $a, b, c \epsilon R$, the following hold if either side is defined (i.e. if one side is defined the other side is also, and they are equal:

i) $a(bc) = (ab)c$

ii) $a(b + c) = ab + ac$

iii) $(a + b)c = ac + bc$

4) Define for $a\epsilon R$, the left multiplicitive set $\mathcal{L}(a) = \{x\epsilon R \mid xa \text{ is defined}\}$, and the right multiplicitive set $\mathcal{R}(a) = \{x\epsilon R \mid ax \text{ is defined}\}$. R satisfies the following conditions:

α. For every $a\epsilon R$, $\mathcal{L}(a) \neq \phi$ and $\mathcal{R}(a) \neq \phi$.

β. If $\mathcal{L}(a) \cap \mathcal{L}(b) \neq \phi$ and $\mathcal{R}(a) \cap \mathcal{R}(b) \neq \phi$, then $a+b$ is defined.

Notation 9 *I will write 0_x to mean the* zero *of the element $x \in R$, i.e. $x + 0_x = x$ holds.*

Lemma 10 *In a ringoid R if $a+b$ is defined, then $\mathcal{L}(a) = \mathcal{L}(b)$ and $\mathcal{R}(a) = \mathcal{R}(b)$.*

Proof. Since $a+b$ is defined we have that $0_a = 0_b$ (this is true because since $a+b$ is defined they are in the same abelian group of morphisms and thus have a common zero element).

Suppose that ra is defined. Then we have that:

$ra = r(a + 0_a) = r(a + 0_b) = r(a + b - b)$.

Thus by definition 8.3.ii above, rb is defined and $\mathcal{L}(a) = \mathcal{L}(b)$ since any element that multiplies to a on the left also multiplies to b on the left.

Similarly we have that $\mathcal{R}(a) = \mathcal{R}(b)$. ■

Example 11 *Given a ringoid R, the set of polynomials $R[x]$ constructed in the usual manner also form a ringoid.*

Example 12 *Rings are of course ringoids.*

Lemma 13 *Let $a, x, r \in R$. Suppose a^2, a^3, ax, xr are defined. Then $(ax)r$ (and thus $a(xr)$) is defined.*

Proof. Since $a \in \mathcal{R}(a) \cap \mathcal{R}(a^2)$ and $a \in \mathcal{L}(a) \cap \mathcal{L}(a^2)$, we have that $\mathcal{R}(a) = \mathcal{R}(a^2)$ and $\mathcal{L}(a) = \mathcal{L}(a^2)$. Thus a^2x is defined. Now consider ax and x. Because $a \in \mathcal{L}(ax) \cap \mathcal{L}(x)$ and $\mathcal{R}(ax) \cap \mathcal{R}(x) \neq \varnothing$ by definition 8.4.α, it follows that $\mathcal{L}(ax) = \mathcal{L}(x)$ and $\mathcal{R}(ax) = \mathcal{R}(x)$. Hence $(ax)r$ is defined. ■

Definition 14 *A subset I of a ringoid R is said to be a* right ideal *in R if:*

$$I - I = \{r - s | r, s \in I, r - s \text{ is defined}\}$$

and

$$IR = \{ir | i \in I, r \in R, ir \text{ is defined}\}$$

are contained in I.

Left ideals and *two-sided* are defined similarly.

Definition 15 *An ideal consisting of zeros alone will be called a* null ideal. *We denote this by N.*

THE FACTOR RINGOID

Definition 16 *Suppose that I is a two sided ideal in the ringoid R. Suppose that $N \subset I$, where N is the null ideal of R. Define the the* factor ringoid R/I *as follows:*

(1) Define an equivalence relation by: $r \equiv s$ iff $r = s + i$ for some $i \in I$. Since I contains all of the zeros of R, $\equiv$ is reflexive. Symmetry holds because if $r = s + i$, then $s = r + (-i)$, and $-i$ exists, since every element in R has an "additive inverse", and is in I since I is a two-sided ideal. Transitivity is shown as follows:

$$\begin{gathered} r \equiv s \Longrightarrow r = s + i \text{ and } s \equiv t \Longrightarrow s = t + j \text{ where } i, j \in I \\ \text{then } r = s + i = (t + j) + i = t + (i + j) \\ \Longrightarrow r \equiv t \text{ since } (i + j) \in I \end{gathered}$$

(2) Define addition and multiplication on some equivalence classes as follows:

$$\begin{gathered} [r] + [s] = [r + s], \text{ if } r + s \text{ is defined} \\ [r]\,[s] = [rs], \text{ if } rs \text{ is defined.} \end{gathered}$$

(3) Check that our definitions are well defined:

- Suppose that $a + b$, $a + i_1$, $b + i_2$ are defined for $a, b \in R;\ \ i_1, i_2 \in I$. The fact that $a + b + i_1 + i_2$ is defined is clear.
- Suppose that ab, $a + i_1$, $b + i_2$ are defined for $a, b \in R;\ \ i_1, i_2 \in I$. We need to show that $(a + i_1)(b + i_2)$ is defined. Since $\mathcal{R}(a) = \mathcal{R}(i_1)$ and ab is defined, we have that $i_1 b$ is defined. Now since ab and $i_1 b$ have a common left and a common right multiplier $ab + i_1 b = (a + i_1) b$ is defined. Similarly we have that $(a + i_1) i_2$ is defined and thus $(a + i_1) b + (a + i_1) i_2 = (a + i_1)(b + i_2)$ is defined.

Some consequences:

1. If J is an ideal in R/S then $\exists$ an ideal I in R such that $J = I/S$.
2. If R is a ringoid with a maximal nilpotent ideal I_N then R/I_N is semi-simple.

Definition 17 *An ideal $I \subset R$, R a ringoid, is called* nilpotent *if there exists an $n \in \mathbb{N}$ such that:*

$$I^n = \left\{ \sum_{finite\ sum} x_1 x_2 \cdots x_n \mid x_i \in I, \text{ and } x_1 x_2 \cdots x_n \text{ is defined} \right\} = \varnothing \text{ or } N.$$

Notation 18 *I will use the notation I_N to denote a nilpotent ideal.*

Definition 19 *We say that a ringoid R satisfies the* minimum condition for right ideals *if any descending chain of right ideals terminates after a finite number of steps.*

Definition 20 *A ringoid R is said to be* semi-simple *if:*

(a) R contains no non-null nilpotent right ideals.
(b) R satisfies the minimum condition for right ideals.

Definition 21 *Let I, J be two right ideals of the ringoid R. Suppose that $\forall r \in R$*

(a)
either $r = i + j, i \in I, j \in J$,
or $r = i, i \in I$,
or $r = j, j \in J$;
and (b) $I \cap J = N$ or $\varnothing$.
Then we say that R is the *direct sum* of I and J and we write $R = I \oplus J$.

Lemma 22 *If I is a minimal right ideal in the ringoid R, then $I^2 = N$ or $\varnothing$; or $I = eR$, $e^2 = e$.*

Lemma 23 *Peirce decomposition:*

Given an idempotent $e \in R$, $R = eR \oplus R'$, where R' is a right ideal.

Proof. Let $r \in R$. Set

$$r = \begin{cases} er + (r - er) & \text{if the right hand side is defined.} \\ r & \text{otherwise.} \end{cases}$$

Let $R' = \{(r - er) \,|r \in R\} \cup \{r| \,(r - er) \text{ is not defined}\}$
Then R' is a right ideal and $eR \cap R' \subset N$. Hence $R = eR \oplus R'$. ■

Theorem 24 *A semi-simple ringoid R is a direct sum of a finite number of minimal right ideals e_iR, $e_i^2 = e_i$.*

Proof. Use the Peirce decomposition successively. Then since R satisfies the minimum condition for right ideals (because it is semi-simple) we can only do this decomposition a finite number of times. Thus we only need to check that a null ideal is not a direct summand. Suppose that N, a null ideal is a direct summand. Let $0 \in N$. We know that there exists an element, call it er, such that $er + 0$ is defined. Then $0 \in eR$. Thus we may omit N as a summand. ■

Lemma 25 *Let $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR \oplus e_{n+1}R \oplus \cdots \oplus e_mR$, such that $e_i^2 = e_i\ \forall i$ and $e_1 = e_i$ is defined iff $j \leq n$. Then:*

1. *e_ie_j is defined for $i, j \leq n$ and e_ie_r is not defined for $i \leq n$ and $r > n$.*
2. *$e_1R \oplus e_2R \oplus \cdots \oplus e_nR$ has a left identity.*
3. *$e_1R \oplus \cdots \oplus e_nR = f_1R \oplus \cdots \oplus f_nR$, for some f_i such that $f_rf_s = \delta_{rs}f_s$ (where δ_{rs} is the Kronecker delta).*
4. *$f_ie_k\,(e_sf_i)$ is defined iff $e_ie_k\,(e_se_i)$ is defined.*

Remark 26 *A semi-simple ringoid can be written as:*

$$R = \left\{e_1^{a_1}R \oplus e_2^{a_1}R \oplus \cdots \oplus e_{a_1}^{a_1}R\right\} \oplus \left\{e_1^{a_2}R \oplus e_2^{a_2}R \oplus \cdots \oplus e_{a_2}^{a_2}R\right\} \oplus \cdots \oplus \left\{e_1^{a_n}R \oplus e_2^{a_n}R \oplus \cdots \oplus e_{a_n}^{a_n}R\right\},$$

such that $e_i^{a_t}$ is not multipliable with $e_j^{a_s}$ for $s \neq t$ and $e_i^{a_s}e_j^{a_s} = \delta_{ij}e_i^{a_s}$ for $s = 1, 2, ..., n\,;\ i, j = 1, 2, ..., a_s$.

Remark 27 *Let R be as in the above remark. Then R satisfies the condition that for every $x \in R$, there exists a unique r such that:*

$$1_x = e_1^{a_r} + e_2^{a_r} + \cdots + e_{a_r}^{a_r} \text{ has the property that } x \cdot 1_x = x.$$

Notation 28 *Instead of writing $e_i^{a_j}R$ in the decomposition of a semi-simple ringoid, I will instead write $e_{ii}^{a_ja_j}R$. This is because we will be dealing with matrices in the following theorems.*

Simple Ringoids

Definition 29 *A ringoid R is said to be simple if:*

1. it satisfies the minimum condition for right ideals,
2. it contains no non-trivial (two-sided) ideals, and
3. $R^2 \neq N, \varnothing$.

Before we continue, let me give a general definition of the *Matrix Ringoid.*

Definition 30 *The* Matrix Ringoid, *M, is defined as follows: Let R be a ring. Let $M_{m,n}(R)$ be the set of all $m \times n$ matrices over R. Let $M = \bigcup_{m,n \in I} M_{m,n}(R)$, where I is any set of natural numbers. Then M is a ringoid.*

Theorem 31 *The matrix ringoid $M = \bigcup_{m,n \in I} M_{m,n}(D)$ contains no proper two-sided ideals if D is a division ring.*

Corollary 32 *Let $M = \bigcup_{m,n \in I} M_{m,n}(D)$, and let D be a division ring, where a_i are not necessarily all different. Suppose addition is defined only for matrices in $M_{a_i,a_j}(D)$. Also suppose multiplication is defined only for*

$$M_{a_i,a_j}(D) \cdot M_{a_j,a_k}(D) \to M_{a_i,a_k}(D).$$

(This allows for matrices of same dimension but of different colours i.e. for $(i,j) \neq (r,s)$, $a_i = a_r$, $a_j = a_s$ but $a_i \times a_j$ matrices not addible to $a_r \times a_s$ matrices and for $j \neq r$, $a_j = a_r$ but $a_i \times a_j$ matrices not multipliable on the left to $a_r \times a_s$ matrices.) Then M contains no proper two-sided ideals.

Corollary 33 *If I is finite then M is simple.*

Definition 34 *Two right ideals I and J in a ringoid R are said to be* R-isomorphic *if there is a one-to-one map $\zeta : I \overset{onto}{\to} J$ such that for $x, y \in I$ and $r \in R$:*

$$\left.\begin{array}{c} \zeta(x+y) = \zeta(x) + \zeta(y) \\ \zeta(xr) = \zeta(x)\, r \end{array}\right\} \text{whenever either side is defined.}$$

Remark 35 *Let $R = I_1 \oplus I_2$ be a direct sum of two right ideals then either $r = i_1 + i_2$, or $r = i_1$, or $r = i_2$.*

Define $\begin{array}{c} \zeta_1(r) = i_1,\, \zeta_2(r) = i_2 \text{ in the first case,} \\ \zeta_1(r) = r,\, \zeta_2(r) = r \text{ in the second and third cases.} \end{array}$
Let $I \subset R$ be a minimal right ideal in R then:

$$\{x \in I \,|\, \zeta_i(x) \text{ is defined}\} \text{ is either } I \text{ or } N.$$

In any case, either $\zeta_i(I) = N$ or $\zeta_i(I)$ is *R-homomorphic* to I.

Lemma 36 *Let I be a right ideal in a ringoid R. Then U, the union of all right ideals R-homomorphic to I, is a two-sided ideal.*

(a) Let R be a simple ringoid, then:

$$R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR,\, e_i^2 = e_i,$$

and all e_iR are R-isomorphic.
(b) Each e_iRe_i is a division ring.

Proof. (of part b only)
By definition 8.3.β it follows that e_iRe_i is an additive group.
By lemma 10 we have: $\begin{array}{c} \mathcal{L}(e_ire_i) = \mathcal{L}(e_ise_i) \\ \mathcal{R}(e_ire_i) = \mathcal{R}(e_ise_i) \end{array}$ for arbitrary $r, s \in R$.
Now $e_ire_i \in \mathcal{L}(e_i)$, and $e_ire_i \in \mathcal{R}(e_i)$, therefore $(e_ire_i)(e_ise_i)$ is defined for arbitrary r and s.
And e_i is the identity for e_iRe_i. Let $0 \neq a \in e_iRe_i$.
$0 \notin aR \subset e_iR \Longrightarrow aR = e_iR \Longrightarrow aRe_i = e_iRe_i$
$\Longrightarrow a = e_iae_i = ae_i \Longrightarrow a(e_iRe_i) = aRe_i = e_iRe_i$
$\Longrightarrow a$ has a right inverse. ∎

Lemma 37 *Let e_1R be R-isomorphic to e_2R then:*

(1) There exist elements e_{12} and e_{21} of R such that:

$$e_1e_{12} = e_{12}, \qquad e_{12}e_2 = e_{12}, \qquad e_{21}e_1 = e_{21}, \qquad e_2e_{21} = e_{21}$$

and

$$e_{12}e_{21} = e_1, \qquad e_{21}e_{12} = e_2,$$

(2) e_1Re_1 is isomorphic to e_2Re_2.

Now here is the *Wedderburn Theorem for Simple Rings:*

Theorem 38 *Every simple ring that is finite-dimensional over a division ring is a matrix ring.*

Now finally the *Wedderburn Theorem for Simple Ringoids*:

Theorem 39 *Let R be a simple ringoid, then R is isomorphic to $\bigcup_{m,n \in I} M_{a_m,a_n}(D)$ with I finite and D a division ring, where addition and multiplication are defined iff the indices m, n fit.*

Corollary 40 *Suppose in a simple ringoid R, x^2 is defined for all $x \in R$, then R is a simple ring.*

Theorem 41 *A semi-simple ringoid R is a direct sum of simple ringoids.*