Finite Math - Fall 2016

Lecture Notes - 9/21/2016

Section 3.4 - Present Value of an Annuity; Amortization

Amortization Schedules. Suppose you are amortizing a debt by making equal payments, but then decided to pay off the debt with one lump-sum payment. How do you find the "pay-off" balance of the debt? (E.g., you take out a 5-year loan with monthly payments for a car, but after 3-years of making payments you decide to just make one final payment to retire the debt.) This "pay-off" is very useful, even if you are not retiring the debt, but refinancing it. When refinancing a debt, you are essentially taking out a new loan to pay-off the previous debt, so you need to know how much unpaid balance remains on the account. When you are making payments into an amortization, at the beginning, a large part of your payment goes towards interest, while later, a larger part goes towards the unpaid balance.

We can see how much of each payment goes towards interest and how much towards unpaid balance by creating an *amortization schedule.*

Example 1. *Construct the amortization schedule for a* \$1,000 *debt that is to be amortized in six equal monthly payments at* 1.25% *interest per month on the unpaid balance.*

Solution. *The first step in this process is to compute the required monthly payment using the amortization formula*

$$PMT = \$1,000\frac{0.0125}{1-(1+0.0125)^{-6}} = \$174.03$$

Now, to figure out how much of the payment goes towards interest and how much towards unpaid balance, we compute the interest due at the end of the first month:

$$\$1,000(0.0125) = \$12.50$$

and so the amount of the payment that goes towards the unpaid balance is:

$$\$174.03 - \$12.50 = \$161.53.$$

Thus, the unpaid balance at the end of the first month is

$$\$1,000 - \$161.53 = \$838.47.$$

To compute the breakdown for the next month, we do the same thing, but with the new unpaid balance. The interest due at the end of month 2*:*

$$\$838.47(0.0125) = \$10.48$$

amount of payment towards unpaid balance:

$$\$174.03 - \$10.48 = \$163.55$$

and so the unpaid balance at the end of 2 *months is*

$$\$838.47 - \$163.55 = \$674.92.$$

Payment Number	Payment	Interest	Unpaid Balance Reduction	Unpaid Balance
0				$\$1,000$
1	$174.03	$12.50	$161.53	$838.47
2	$174.03	$10.48	$163.55	$674.92
3	$174.03	$8.44	$165.59	$509.33
4	$174.03	$6.37	$167.66	$341.67
5	$174.03	$4.27	$169.76	$171.91
6	$174.03	$2.15	$171.91	$0.00
Total	$\$1,044.21$	$44.21	$\$1,000$	

Section 4.1 - Systems of Linear Equations in Two Variables

Suppose we go to a movie theater and there are two packages for discounted tickets:
Package 1: 2 adult tickets and 1 child ticket for $32
Package 2: 1 adult ticket and 3 child tickets for $36

Based off of this information, can we figure how much the adult and child ticket discount prices are?

We can! To do this let A stand for the price of the adult ticket and let C stand for the price of the child ticket, then we get the following two equations from the two packages:

$$\begin{array}{rcrcl} 2A & + & C & = & 32 \\ A & + & 3C & = & 36 \end{array}$$

This is a system of two linear equations in two variables. To find the answer, we need to find a pair of numbers (A, C) which satisfy *both* equations simultaneously.

Definition 1 (System of Two Linear Equations in Two Variables)**.** *Given the* linear system

$$\begin{array}{rcrcl} ax & + & by & = & h \\ cx & + & dy & = & k \end{array}$$

where a, b, c, d, h, and k are real constants, a pair of numbers $x = x_0$ and $y = y_0$ (often written as an ordered pair (x_0, y_0)) is a solution *of this system if each equation is satisfied by the pair. The set of all such ordered pairs is called the* solution set *for the system. To* solve *a system is to find its solution set.*

There are a few ways we can go about solving this: *graphically*, using *substitution*, and *elimination by addition*.

Solving by Graphing. To solve this problem by graphing, we simply graph the two equations in the system, then find the intersection

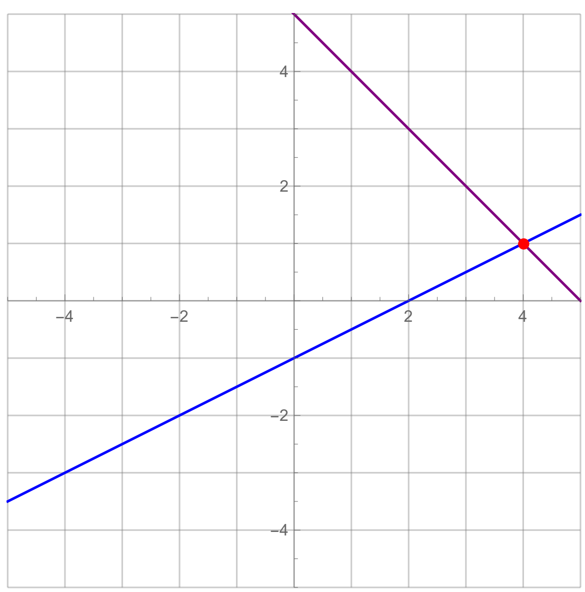
The blue line is the graph of $2A + C = 32$ and the purple line is the graph of $A + 3C = 36$. The red point is the intersection point $(12, 8)$. So the ticket prices are \$12 for an adult ticket and \$8 for a child ticket.

There are actually 3 types of solutions to a system of linear equations

(1) Consider the system

$$\begin{array}{rcrcl} x & - & 2y & = & 2 \\ x & + & y & = & 5 \end{array}$$

If we graph the lines, we get

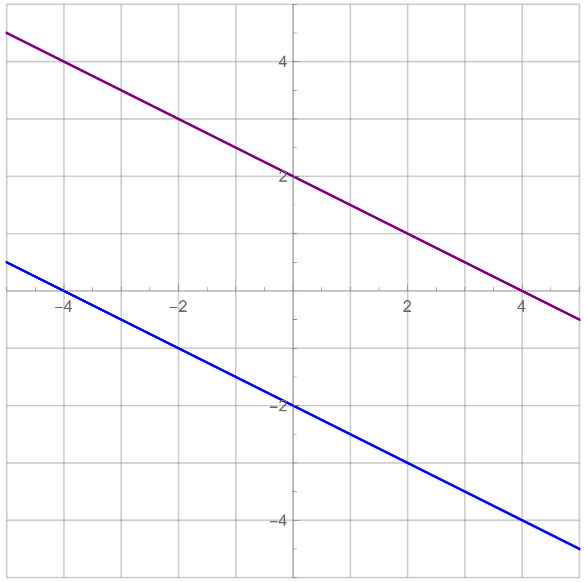


In this case, like before, we see only the *one solution* at $(4, 1)$.

(2) Consider the system

$$\begin{array}{rcrcl} x & + & 2y & = & 4 \\ 2x & + & 4y & = & 8 \end{array}$$

If we graph the lines, we get

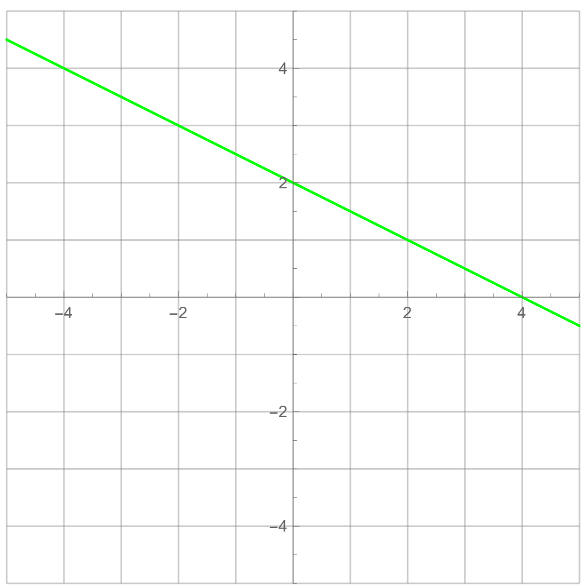


In this case, the lines are parallel and so they never intersect. In this case, there is *no solution*.

(3) Consider the system

$$\begin{array}{rcrcr} 2x & + & 4y & = & 8 \\ x & + & 2y & = & 4 \end{array}$$

If we graph the lines, we get



Here, both of the lines are exactly the same. In this case, there is an infinite number of solutions.

Definition 2. *A system of linear equations is called* consistent *if it has one or more solutions and* inconsistent *if it has no solutions. Further, a consistent system is called* independent *if it has exactly one solution (called the* unique solution*) and is called* dependent *if it has more than one solution. Two systems of equations are called* equivalent *if they have the same solution set.*

Theorem 1. *The linear system*

$$\begin{array}{rcrcl} ax & + & by & = & h \\ cx & + & dy & = & k \end{array}$$

must have

(1) *Exactly one solution (consistentent and independent).*

(2) *No solution (inconsistent).*

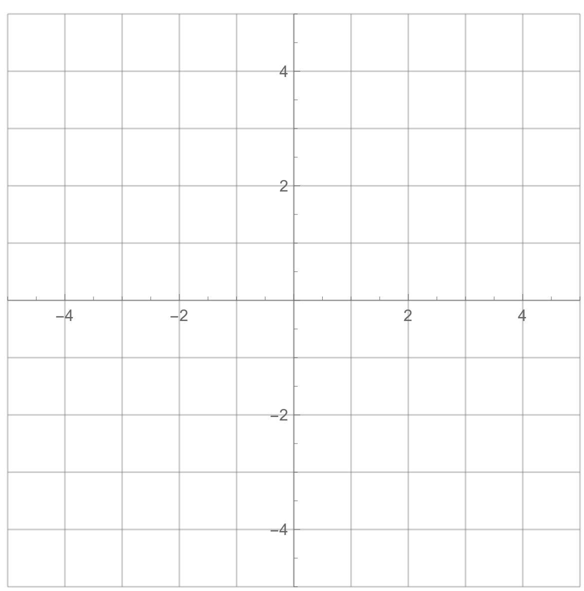
(3) *Infinitely many solutions (consistent and dependent).*

There are no other possibilities.

Example 2. *Solve the following systems of equations using the graphing method. Determine whether there is one solution, no solutions, or infinitely many solutions. If there is one solution, give the solution.*

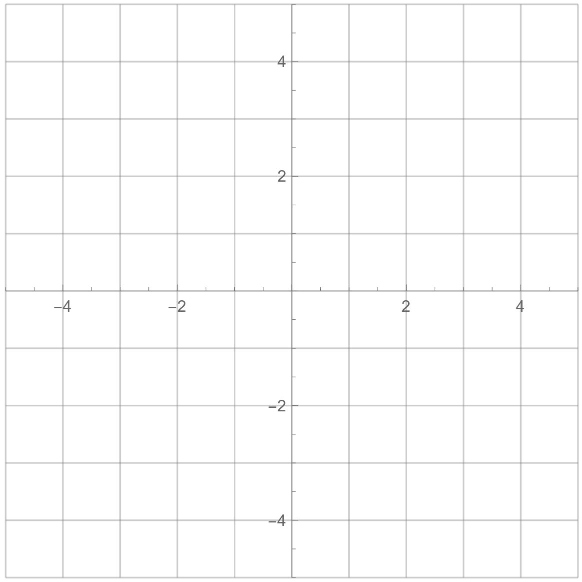
(a)

$$\begin{array}{rcrcl} x & + & y & = & 4 \\ 2x & - & y & = & 5 \end{array}$$



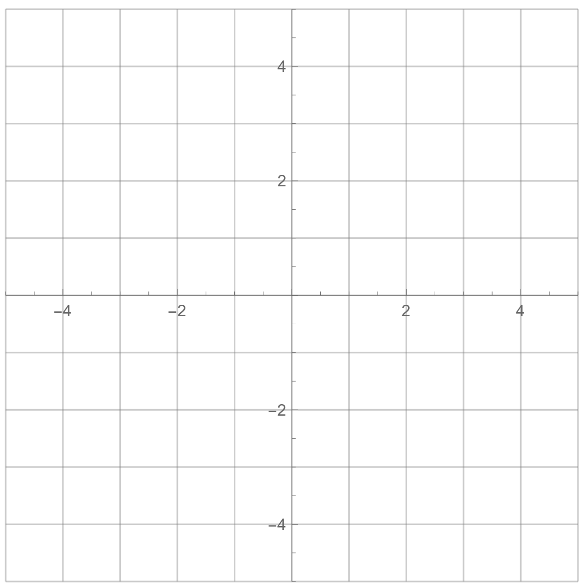
(b)

$$\begin{array}{rcrcl} 6x & - & 3y & = & 9 \\ 2x & - & y & = & 3 \end{array}$$



(c)

$$\begin{array}{rcrcr} 2x & - & y & = & 4 \\ 6x & - & 3y & = & -18 \end{array}$$



Solving by Substitution. When solving a system by substitution, we solve for one of the variables in one of the equations, then plug that variable into the other equation.

Example 3. *Solve the following system using substitution*

$$\begin{array}{rcrcr} 2x & - & y & = & 3 \\ x & + & 2y & = & 14 \end{array}$$

Solution. *Let's solve the first equation for y. To do this, we'll move the y to the right side, and the 3 to the left:*

$$2x - 3 = y$$

Then we plug this into the second equation for y:

$$x + 2(2x - 3) = 14$$

then we solve for x in this

$$x + 4x - 6 = 5x - 6 = 14$$

which gives

$$5x = 20$$

and so

$$x = 4.$$

Then we plug this into the equation we have for y to find that

$$y = 2(4) - 3 = 8 - 3 = 5$$

And so the solutions is $x = 4, y = 5$.

Example 4. *Solve the following system using substitution*

$$\begin{array}{rcrcr} 3x & + & 2y & = & -2 \\ 2x & - & y & = & -6 \end{array}$$

Solution. $x = -2, y = 2$

Solving Using Elimination. We now turn to a method that, unlike graphing and substitution, is generalizable to systems with more than two variables easily. There are a set of rules to follow when doing this

Theorem 2. *A system of linear equations is transformed into an equivalent system if*

(1) *two equations are interchanged*

(2) *an equation is multiplied by a nonzero constant*

(3) *a constant multiple of one equation is added to another equation.*

Example 5. *Solve the following system using elimination*

$$\begin{array}{rcrcr} 3x & - & 2y & = & 8 \\ 2x & + & 5y & = & -1 \end{array}$$

Solution. *If we subtract the second equation from the first one, we end up with the new system*

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ 2x & + & 5y & = & -1 \end{array}$$

Now, we can subtract 2 times the first equation ($2x - 14y = 18$) from the second equation to get

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ & & 19y & = & -19 \end{array}$$

Now we divide the second equation by 19 to get

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ & & y & = & -1 \end{array}$$

and finally, we will add 7 times the second equation ($7y = -7$) to the first equation

$$\begin{array}{rcrcr} x & & & = & 2 \\ & & y & = & -1 \end{array}$$

This gives the answer of $x = 2, y = -1$.

We could have also used a combination of substitution and elimination above, for example, once we knew that $y = -1$, we could have just plugged that into the first equation, but this solution was a little preview for the later sections.

Example 6. *Solve the system using elimination*

$$\begin{array}{rcrcr} 5x & - & 2y & = & 12 \\ 2x & + & 3y & = & 1 \end{array}$$

Solution. $x = 2, y = -1$

We now want to look at the case when the system does not have one unique solution, but is either inconsistent or is consistent but dependent.

Example 7. *Solve the system*

$$\begin{array}{rcrcr} 2x & + & 6y & = & -3 \\ x & + & 3y & = & 2 \end{array}$$

Solution. *We begin by subtracting twice the second equation from the first*

$$\begin{array}{rcrcr} & & 0 & = & -7 \\ x & + & 3y & = & 2 \end{array}$$

The first equation has become $0 = -7$ which is obviously untrue. This is an example of what happens when the system is inconsistent*.*

Example 8. *Solve the system*

$$\begin{array}{rcrcr} x & - & \frac{1}{2}y & = & 4 \\ -2x & + & y & = & -8 \end{array}$$

Solution. *First, let's multiply the first equation by 2 to get rid of the fraction*

$$\begin{array}{rcrcr} 2x & - & y & = & 8 \\ -2x & + & y & = & -8 \end{array}$$

Now, if we add the two equations together, we get

$$0 = 0$$

which is always true. This means that the two equations are the same equation, just one is (maybe) multiplied by a constant. This is a consistent but dependent *system of equations. If we let $x = k$, where k is any real number, then we get that $y = 2k - 8$. So, for any k, $(k, 2k - 8)$ is a solution. In this case, the variable k is called a* parameter*.*

Example 9. *Solve the systems*

(a)

$$\begin{array}{rcrcr} 5x & + & 4y & = & 4 \\ 10x & + & 8y & = & 4 \end{array}$$

(b)

$$\begin{array}{rcrcr} 6x & - & 5y & = & 10 \\ -12x & + & 10y & = & -20 \end{array}$$