Finite Math - Fall 2016

Lecture Notes - 10/19/2016

Section 4.4 - Matrices: Basic Operations

Addition and Subtraction. First, let's define what it means for two matrices to be equal.

Definition 1 (Equal). *Two matrices are* equal *if they are the same size and the corresponding elements in each matrix are equal.*

For example, the equality

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

is true if and only if

$$\begin{matrix} a = u & b = v \\ c = w & d = x \\ e = y & f = z \end{matrix}.$$

In order to add or subtract matrices **they must be the same size**.

- When adding matrices, we just add the corresponding elements.
- When subtracting matrices, we just subtract the corresponding elements.

Example 1. *Find the indicated operations*

(a)

$$\begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 3 & 4 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 & 2 & -1 \\ -1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$$

Solution.

(a)

$$\begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 3+(-2) & 2+3 \\ -1+1 & -1+(-1) \\ 0+2 & 3+(-2) \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & -2 \\ 2 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3-2 & 2-(-2) \\ 5-3 & 0-4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & -4 \end{bmatrix}$$

(c) *These matrices are not the same size and so cannot be added.*

Example 2. *Find the indicated operations*

(a)

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 & -2 \end{bmatrix}$$

Scalar Multiplication. If k is a number and M is a matrix, we can form the scalar product kM by just multiplying every element of M by k.

Example 3. *Find*

$$-2\begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix}$$

Solution.

$$-2\begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} -2(3) & -2(-1) & -2(0) \\ -2(-2) & -2(1) & -2(3) \\ -2(0) & -2(-1) & -2(-2) \end{bmatrix} = \begin{bmatrix} -6 & 2 & 0 \\ 4 & -2 & -6 \\ 0 & 2 & 4 \end{bmatrix}$$

Example 4. *Find*

$$5\begin{bmatrix} 1 & -1 \\ 0 & -2 \\ 2 & -3 \\ 3 & 3 \end{bmatrix}$$

Matrix Multiplication. In order to define matrix multiplication, it is easier to first define the product of a row matrix with a column matrix.

Definition 2. *Suppose we have a $1 \times n$ row matrix A and an $n \times 1$ column matrix B where*

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

It is very important that the number of columns in A matches the number of rows in B.

Example 5. *Find*

$$\begin{bmatrix} -1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} -1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix} = (-1)(2)+(0)(3)+(3)(4)+(2)(-1) = -2+0+12-2 = 8$$

Example 6. *Find*

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Definition 3 (Matrix Multiplication)**.** *Let A be an $m \times p$ matrix and let B be a $p \times n$ matrix. Let R_i denote the matrix formed by the i^{th} row of A and let C_j denote the matrix formed by the j^{th} column of B. Then the ij^{th} element of the matrix product AB is $R_i C_j$.*

Remark 1. *It is very important that the number of columns of A matches the number of rows of B, otherwise the products $R_i C_j$ would not be able to be defined. That is, if A is an $m \times n$ matrix and B is an $p \times q$ matrix, the product AB is defined if and only if $n = p$.*

Example 7. *Let* $A = \begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$. *Find the following products, if possible.*

(a) AB

(b) BA

(c) CD

(d) DC

(e) CB

(f) D^2

Solution.

(a) *Since* A *is* 2×4 *and* B *is* 3×2, *the product* AB *is not defined.*

(b)

$$BA = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 & R_1C_4 \\ R_2C_1 & R_2C_2 & R_2C_3 & R_2C_4 \\ R_3C_1 & R_3C_2 & R_3C_3 & R_3C_4 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} & \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-1)+(1)(1) & (-1)(0)+(1)(2) & (-1)(3)+(1)(2) & (-1)(-2)+(1)(0) \\ (2)(-1)+(3)(1) & (2)(0)+(3)(2) & (2)(3)+(3)(2) & (2)(-2)+(3)(0) \\ (1)(-1)+(0)(1) & (1)(0)+(0)(2) & (1)(3)+(0)(2) & (1)(-2)+(0)(0) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & -1 & 2 \\ 1 & 6 & 12 & -4 \\ -1 & 0 & 3 & -2 \end{bmatrix}$$

(c)

$$CD = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(-2)+(2)(1) & (1)(4)+(2)(-2) \\ (-1)(-2)+(-2)(1) & (-1)(4)+(-2)(-2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution. (d) $\begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}$

(e) *Not defined.*

(f) $\begin{bmatrix} 8 & -16 \\ -4 & 8 \end{bmatrix}$

Remark 2. *Note that parts (c) and (d) show that matrix multiplication is not commutative. That is, it is not necessarily true that $AB = BA$ for matrices A and B, even if both matrix products are defined.*

Example 8. *Find a, b, c, and d such that*

$$\begin{bmatrix} 6 & -5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -16 & 64 \\ 24 & -6 \end{bmatrix}$$

Solution. *If we multiply out the matrices on the left, we get the equation*

$$\begin{bmatrix} 6a - 5c & 6b - 5d \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} -16 & 64 \\ 24 & -6 \end{bmatrix}$$

And so we have the following two systems of equations

$$\begin{aligned} 6a - 5c &= -16 \\ 3c &= 24 \end{aligned}$$

and

$$\begin{aligned} 6b - 5d &= 64 \\ 3d &= -6 \end{aligned}$$

The augmented matrices for these two systems are

$$\left[\begin{array}{cc:c} 6 & -5 & -16 \\ 0 & 3 & 24 \end{array}\right] \quad \textit{and} \quad \left[\begin{array}{cc:c} 6 & -5 & 64 \\ 0 & 3 & -6 \end{array}\right]$$

Notice that these two systems have the same coefficient matrix! In the first augmented matrix, we are aiming to solve for the variables a and c which are the first column of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and in the second augmented matrix, we are solving for the variables b and d. Because of this, we can just stick the augmented matrices together to form the new one

$$\left[\begin{array}{cc:cc} 6 & -5 & -16 & 64 \\ 0 & 3 & 24 & -6 \end{array}\right]$$

Then, as before, we aim to get a reduced form on the left side, and that will simultaneously solve both systems. That is, we are aiming for

$$\left[\begin{array}{cc:cc} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}\right]$$

So, let's solve the system

$$\left[\begin{array}{cc:cc} 6 & -5 & -16 & 64 \\ 0 & 3 & 24 & -6 \end{array}\right] \overset{\frac{1}{6}R_1 \to R_1}{\sim} \left[\begin{array}{cc:cc} 1 & -\frac{5}{6} & -\frac{16}{6} & \frac{64}{6} \\ 0 & 3 & 24 & -6 \end{array}\right] \overset{\frac{1}{3}R_2 \to R_2}{\sim} \left[\begin{array}{cc:cc} 1 & -\frac{5}{6} & -\frac{16}{6} & \frac{64}{6} \\ 0 & 1 & 8 & -2 \end{array}\right]$$

$$\overset{R_1 + \frac{5}{6}R_2 \to R_1}{\sim} \left[\begin{array}{cc:cc} 1 & 0 & 4 & 9 \\ 0 & 1 & 8 & -2 \end{array}\right]$$

So, we get that $a = 4$, $b = 9$, $c = 8$, and $d = -2$.

Example 9. *Find a, b, c, and d such that*

$$\begin{bmatrix} 6 & -5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{5}{18} \\ 0 & \frac{1}{3} \end{bmatrix}$$