Finite Math - Fall 2016

Lecture Notes - 10/31/2016

Section 4.6 - Matrix Equations and Systems of Linear Equations

Matrix Equations.

Theorem 1. *Assume that all products and sums are defined for the indicated matrices A, B, C, I, and 0 (where 0 stands for the zero matrix). Then*

- *Addition Properties*

  (1) *Associative*

  $$(A+B)+C = A+(B+C)$$

  (2) *Commutative*

  $$A+B = B+A$$

  (3) *Additive Identity*

  $$A+0 = 0+A = A$$

  (4) *Additive Inverse*

  $$A+(-A) = (-A)+A = 0$$

- *Multiplication Properties*

  (1) *Associative Property*

  $$A(BC) = (AB)C$$

  (2) *Multiplicative Identity*

  $$AI = IA = A$$

  (3) *Multiplicative Inverse*
  If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$

- *Combined Properties*

  (1) *Left Distributive*

  $$A(B+C) = AB+AC$$

  (2) *Right Distributive*

  $$(B+C)A = BA+CA$$

- *Equality*

  (1) *Addition*
  If $A = B$*, then* $A + C = B + C$

  (2) *Left Multiplication*
  If $A = B$*, then* $CA = CB$

  (3) *Right Multiplication*
  If $A = B$*, then* $AC = BC$

We can use the rules above to solve various matrix equations. In the next 3 examples, we will assume all necessary inverses exists.

Example 1. *Suppose* A *is an* $n \times n$ *matrix and* B *and* X *are* $n \times 1$ *column matrices. Solve the matrix equation for* X

$$AX = B.$$

Solution. *If we multiply both sides of this equation ON THE LEFT by* A^{-1} *we find*

$$A^{-1}(AX) = A^{-1}B \quad \Longrightarrow \quad (A^{-1}A)X = IX = X = A^{-1}B$$

Example 2. *Suppose* A *is an* $n \times n$ *matrix and* B*,* C*, and* X *are* $n \times 1$ *matrices. Solve the matrix equation for* X

$$AX + C = B.$$

Solution. *Begin by subtracting* C *to the other side*

$$AX + C = B \quad \Longrightarrow \quad AX = B - C$$

and now multiply on the left by A^{-1}

$$A^{-1}(AX) = A^{-1}(B-C) \quad \Longrightarrow \quad (A^{-1}A)X = IX = X = A^{-1}(B-C) = A^{-1}B - A^{-1}C$$

Example 3. *Suppose* A *and* B *are* $n \times n$ *matrices and* C *is an* $n \times 1$ *matrix. Solve the matrix equation for* X

$$AX - BX = C.$$

What size matrix is X*?*

Matrix Equations and Systems of Linear Equations. We can also solve systems of equations using the above ideas. These apply in the case that the system has the same number of variables as equations and the coefficient matrix of the system is invertible. If that is the case, for the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & a_{1n}x_n & = & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

We can create the matrix equation

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{1n} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then, if A is invertible (as is the case when the system is consistent and independent, i.e., exactly one solution), we have

$$X = A^{-1}B.$$

Example 4. *Solve the system of equations using matrix methods*

$$\begin{array}{ccccc} x & + & 2y & = & 1 \\ x & + & 3y & = & 3 \end{array}$$

Solution. *Begin by writing this system as a matrix equation*

$$AX = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = B$$

Our goal is to find $A^{-1}B$, so first find A^{-1}:

$$A^{-1} = \frac{1}{3-2} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

So we have $x = -3$ and $y = 2$.

Example 5. *Solve the system of equations using matrix methods*

$$\begin{array}{ccccc} 2x & + & y & = & 1 \\ 5x & + & 3y & = & -1 \end{array}$$

Solution. $x = 4$, $y = -7$

Section 5.1 - Linear Inequalities in Two Variables

Graphing Linear Inequalities in Two Variables. There are 4 types of linear inequalities

$$Ax + By \geq C \qquad Ax + By > C$$

$$Ax + By \leq C \qquad Ax + By < C$$

There is a simple procedure to graphing any of these. If equality is not allowed in an inequality, we call it a *strict inequality*, otherwise we simply call it an inequality.

Procedure. (1) *Graph the line $Ax + By = C$ as a dashed line if the inequality is strict. Otherwise, graph it as a solid line.*

(2) *Choose a test point anywhere in the plane, as long as it is not the line. (The origin, $(0, 0)$ is often an easy choice here, but if it is on the line, $(1, 0)$ or $(0, 1)$ are also easy points to check.)*

(3) *Plug the point from step (2) into the inequality. Is the inequality true? Shade in the side of the line with that point. If the inequality is false, shade in the other side.*

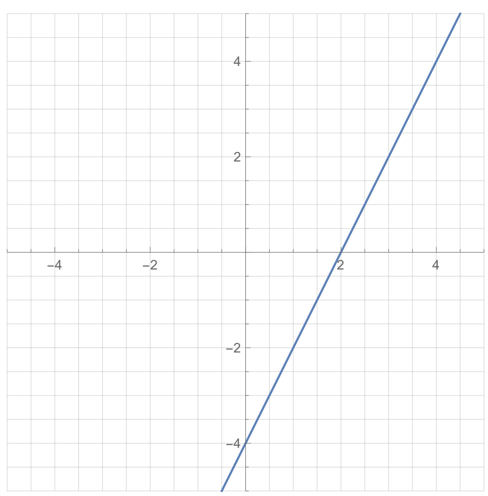
Example 6. *Graph the inequality*

$$6x - 3y \geq 12$$

Solution. *The line we want to graph is*

$$6x - 3y = 12 \quad or \quad y = 2x - 4.$$

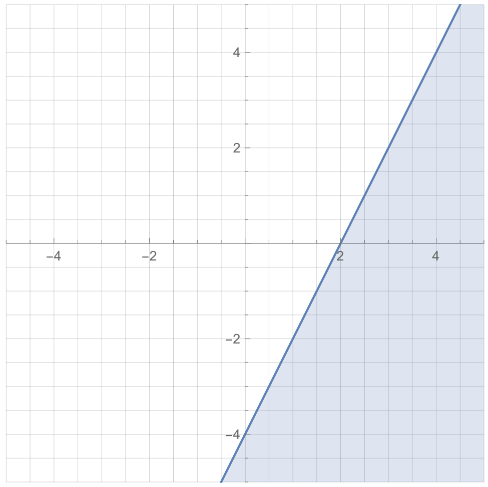
Since the inequality is not strict, we graph it with a solid line.



The point $(0, 0)$ is not on the line, so we check that point in the inequality

$$6(0) - 3(0) = 0 \geq 12$$

This is false, so we shade in the side of the line without the origin.



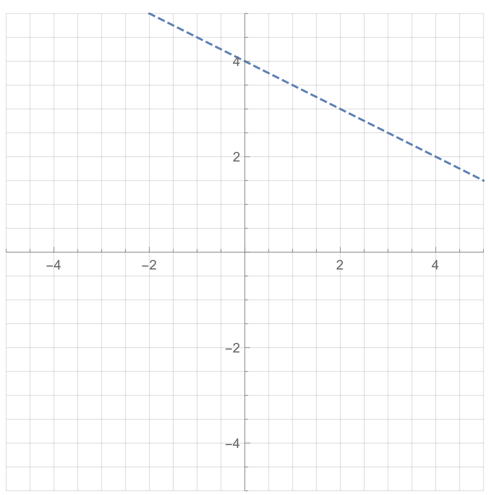
Example 7. *Graph the inequality*

$$4x + 8y < 32$$

Solution. *The line we want to graph is*

$$4x + 8y = 32 \quad \textit{or} \quad y = -\frac{1}{2}x + 4.$$

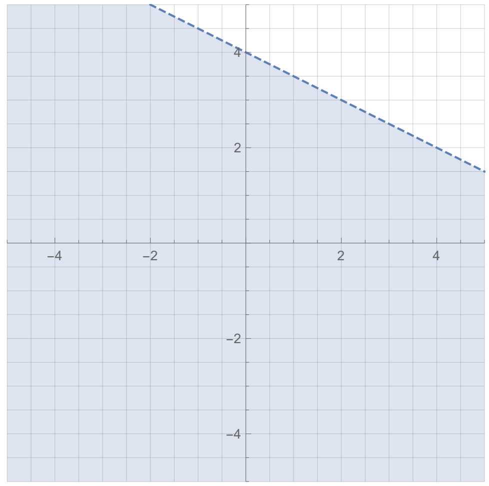
Since the inequality is strict, we graph it with a dashed line.



The point $(0, 0)$ *is not on the line, so we check that point in the inequality*

$$4(0) + 8(0) = 0 < 32$$

This is true, so we shade in the side of the line with the origin.



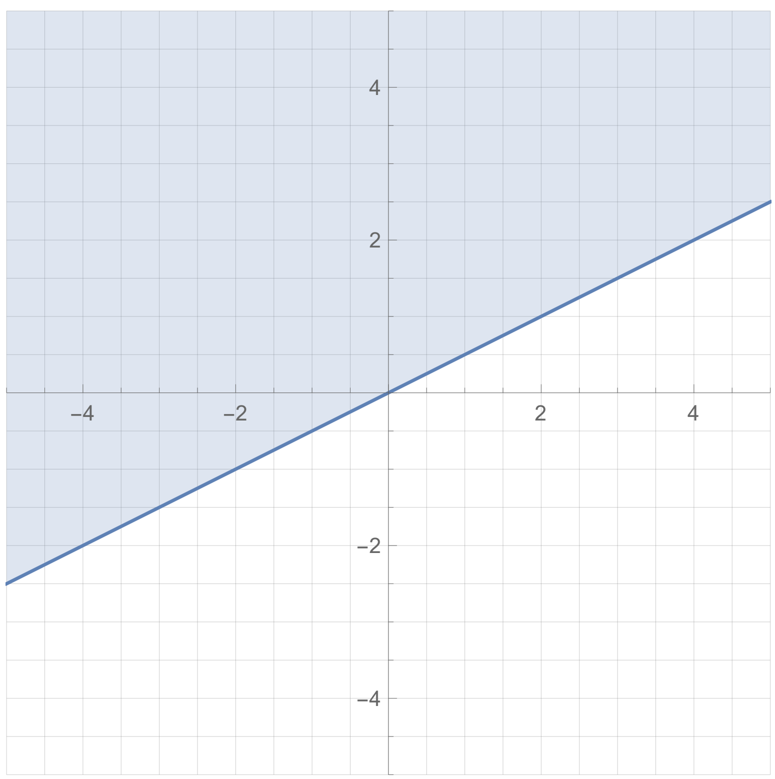
Example 8. *Graph the inequality*

$$2y \leq 10$$

Example 9. *Graph the inequality*

$$2x - 5y > 10$$

Example 10. *Consider the graphed region below.*



(a) *Find an equation for the boundary of the region in the form* $Ax + By = C$.

(b) *Find a linear equality which describes this region.*

Solution.

(a) *Observing the graph, we see that the boundary line passes through* $(0, 0)$ *and* $(2, 1)$. *Using the point-slope form, we get*

$$y - 0 = \frac{1-0}{2-0}(x - 0)$$

which simplifies to

$$y = \frac{1}{2}x$$

Putting this in the required form gives

$$x - 2y = 0.$$

(b) *Because the boundary line is solid, we are going to replace* $=$ *with either* $\geq$ *or* $\leq$. *To figure out which one, we pick a test point which is not on the line and choose the inequality appropriately. If the test point comes from the shaded region, then we pick the inequality which makes the statement true. If the test point comes from outside the shaded region, pick the inequality which makes the statement false.*

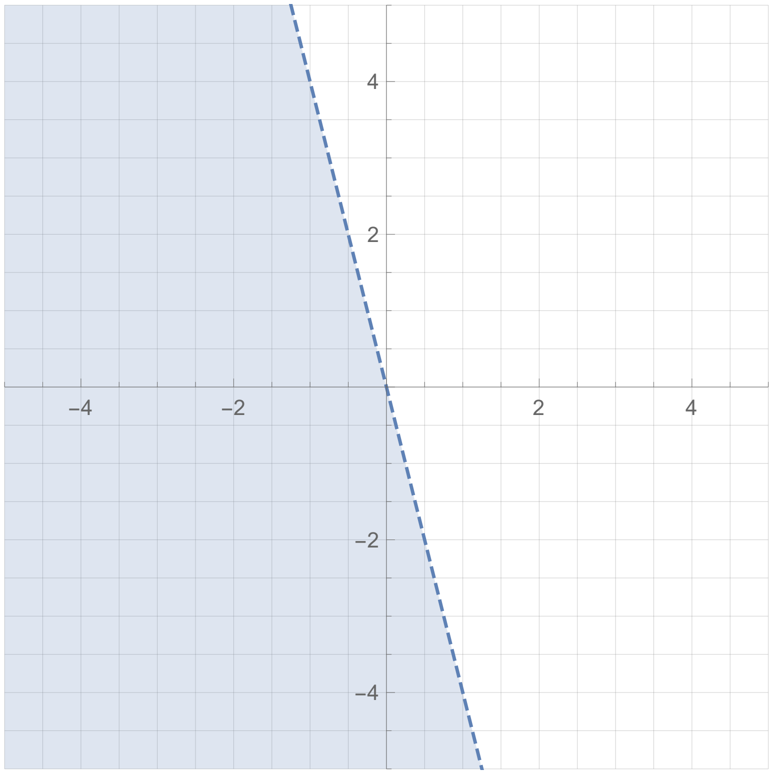
Since $(0, 0)$ *actually is on this line, we will pick* $(1, 0)$ *as our test point. Notice that* $(1, 0)$ *is* outside *the shaded region. Plugging* $(1, 0)$ *into the equation gives*

$$x - 2y = 1 - 2(0) = 1 \boxed{?} 0.$$

Since $(1, 0)$ *is not in the shaded region, we need to pick the one of* $\geq$ *and* $\leq$ *to replace* $\boxed{?}$ *which makes the statement false. This means we choose* $\leq$ *giving that the inequality for this picture is*

$$x - 2y \leq 0.$$

Example 11. *Consider the graphed region below.*



(a) *Find an equation for the boundary of the region in the form $Ax + By = C$.*

(b) *Find a linear equality which describes this region.*

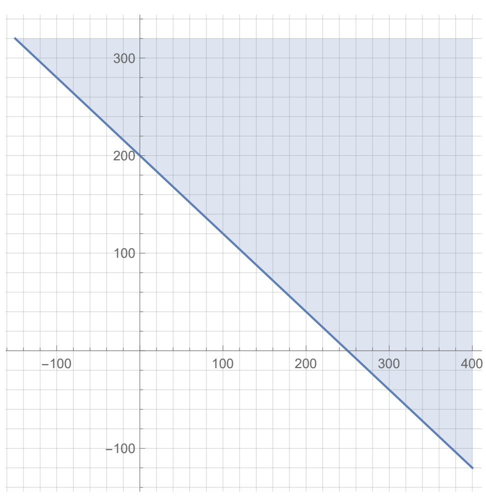
Application.

Example 12. *A food vendor at a rock concert sells hot dogs for \$4 and hamburgers for \$5. How many of these sandwiches must be sold to produce sales of at least \$1,000? Express the answer as a linear equality and sketch its graph.*

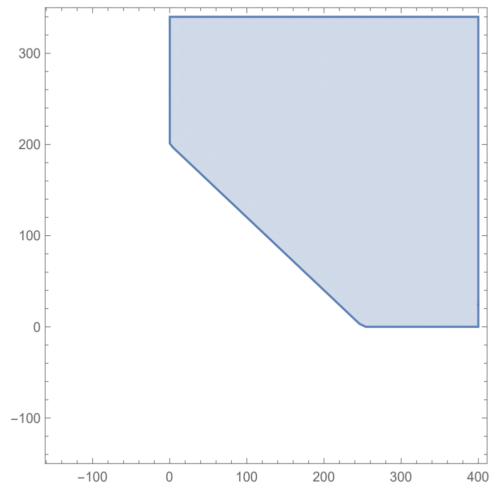
Solution. *Suppose the vendor sells x hot dogs and y hamburgers. Then the seller has made $4x + 5y$ dollars. The sellers wants to make at least \$1000, so we get*

$$4x + 5y \geq 1000.$$

If we graph this we get



But since a negative number of hot dogs or hamburgers cannot be sold, we also have the inequalities $x \geq 0$ *and* $y \geq 0$ *to add to this which gives the graph*



The solution is then

$$\begin{cases} 4x + 5y \geq 1000 \\ x \geq 0, \;\; y \geq 0 \end{cases}$$