Finite Math - J-term 2017

Lecture Notes - 1/3/2017

Homework

- Section 1.2 - 5, 6, 7, 8, 23, 25, 27
- Graph the following lines:
  (1) $y = 2x$
  (2) $2x + 3y = 0$
  (3) $5x - 6y = 0$
- Section 2.5 - 1, 3, 5, 31, 34, 59, 60, 62
- Section 2.6 - 13, 16, 18, 20, 27, 30, 32, 43, 44, 47, 92
- Chapter 2 Review - 96

Section 1.2 - Graphs and Lines

Definition 1 (Line)**.** *A* line *is the graph of an equation of the form*

$$Ax + By = C$$

where not both of A and B are equal to zero (i.e., if $A = 0$, then $B \neq 0$ and vice-versa).

Graphing Lines. There are two common ways of graphing lines: by **finding intercepts** and by **using the slope and a point**. We will focus on the method of finding intercepts here in the notes. You can read about using the slope to graph a line in the textbook.

Finding Intercepts.

Definition 2 (Intercept)**.** *A point of the form $(a, 0)$ on a line is called an x-*intercept *and a point of the form $(0, b)$ is called a y-*intercept*.*

Every line will have at least one intercept, but most have two. There are three special cases in which the line has only one intercept: if $A = 0$, $B = 0$, or $C = 0$. We will return to these special cases in a little bit.

Assume the line $Ax + By = C$ has both an x- and $y-$ intercept, we find them as follows:

- To find the x-intercept, we set $y = 0$ in the equation of the line and solve for x. Symbolically, this means that

$$x = \frac{C}{A}.$$

- To find the y-intercept, we set $x = 0$ in the equation of the line and solve for y. Symbolically, this means that

$$y = \frac{C}{B}.$$

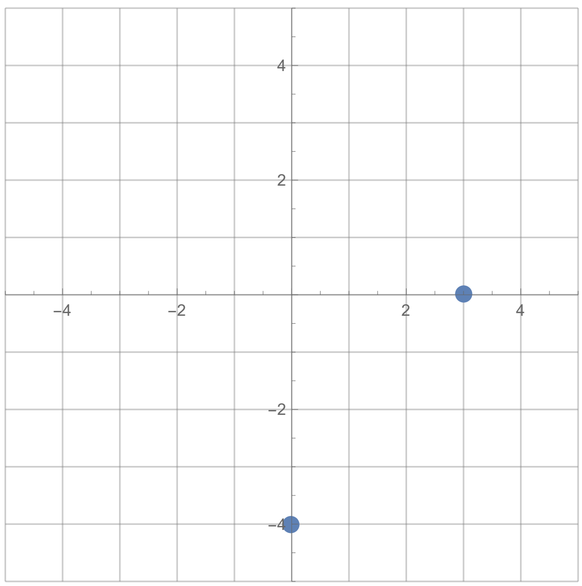
To graph a line using intercepts, we plot the two intercepts in the xy-plane, and draw a line through the points:

Example 1. *Graph the line $4x - 3y = 12$ using intercepts.*

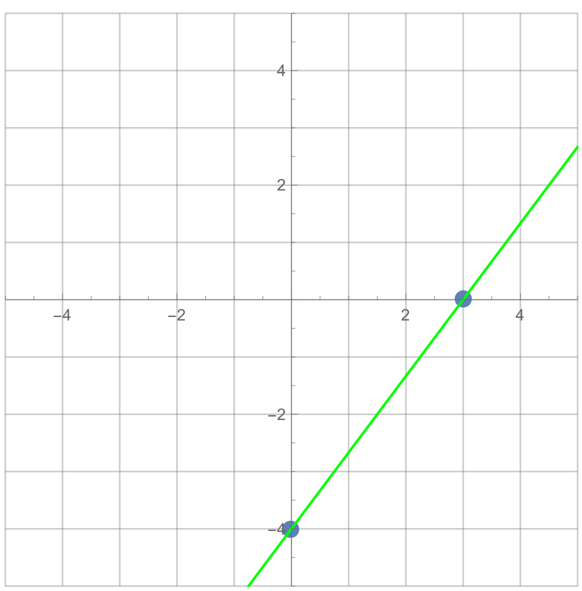
Solution. *First find the intercepts:*

- *set $x = 0$ to get $-3y = 12$ so $y = -4$. Thus the y-intercept is $(0, -4)$.*
- *set $y = 0$ to get $4y = 12$ so $x = 3$. Thus the x-intercept is $(3, 0)$.*

Now plot these points:



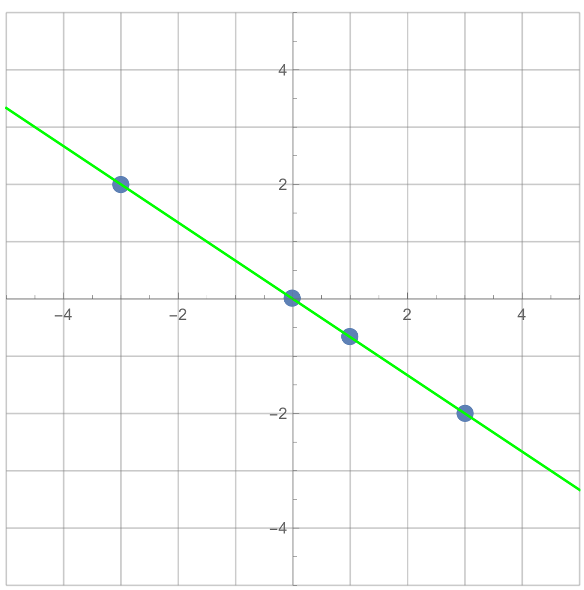
Then we draw a line through them:



Now, let's talk about one of those special cases, when $C = 0$ in $Ax + By = C$. We will assume that $A, B \neq 0$ here. If $C = 0$, you'll find that solving for the x-intercept as above gives $(0, 0)$ and solving for the y-intercept also gives $(0, 0)$. This means that both the x- and y- intercepts are at the origin. So, to graph the line $Ax + By = 0$, we need to come up with another point. You can really just pick any number other than 0 for x or y, then solve for the opposite. One easy thing that always works is to use one of the points $(B, -A)$ or $(-B, A)$, both are points on the line (check this!). That is, you just take the coefficients of x and y, flip their order, and put a minus sign in front of one of them. As a matter of fact, you could just use the two points $(B, -A)$ and $(-B, A)$ to graph the line. There's many other points you could use (which might be simpler than the previous two), but these always work.

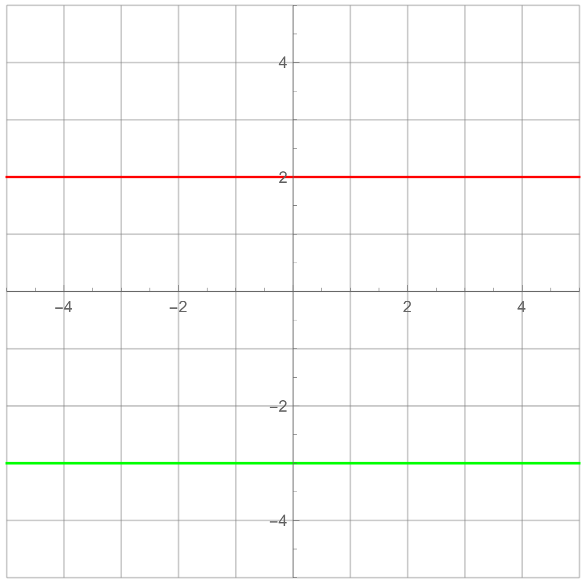
Example 2. *Graph the line* $2x + 3y = 0$.

Solution. *If we try to solve for either intercept on this line, we will get the point* $(0, 0)$. *We need to come up with a second point now to actually graph the line... We can use the trick from above to come up with the point* $(-3, 2)$ *or* $(3, -2)$. *An alternate way we could find a point is, say, to set* $x = 1$. *Then the equation for the line gives us* $2(1) + 3y = 2 + 3y = 0$, *and solving for* y *gives* $y = -\frac{2}{3}$. *So, we also have the point* $\left(1, -\frac{2}{3}\right)$ *on the line. We'll plot all of these points, just so we can see that they are all on the line, but we could get away with just any two of them.*

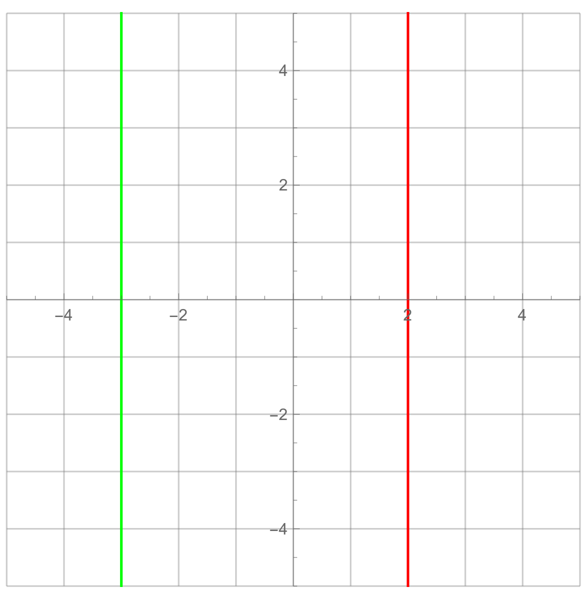


Horizontal and Vertical Lines. The cases when $A = 0$ or $B = 0$ in $Ax + By = C$ correspond to horizontal and vertical lines, respectively.

- If $A = 0$, we end up with the line $y = \dfrac{C}{B}$, which is a horizontal line where every y-value is $\dfrac{C}{B}$. A special one of these is when C is also zero so we get the equation $y = 0$. The graph of this line is the x-axis. Here are the graphs of $y = 2$ (red) and $y = -3$ (green).



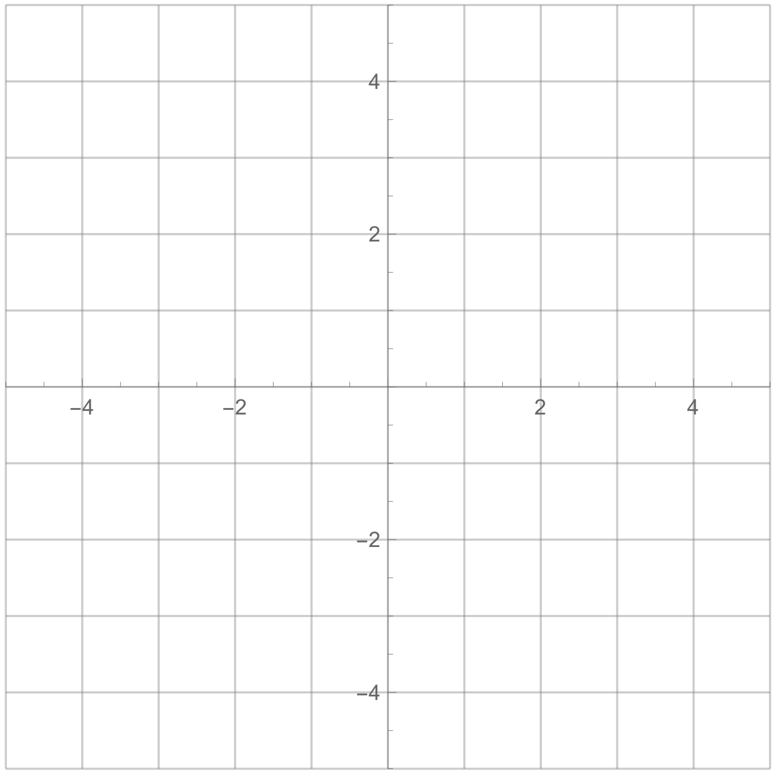
- If $B = 0$, we end up with the line $x = \dfrac{C}{A}$, which is a vertical line where every x-value is $\dfrac{C}{A}$. A special one of these is when C is also zero so we get the equation $x = 0$. The graph of this line is the y-axis. Here are the graphs of $x = 2$ (red) and $x = -3$ (green).



Example 3. *Graph the following lines:*

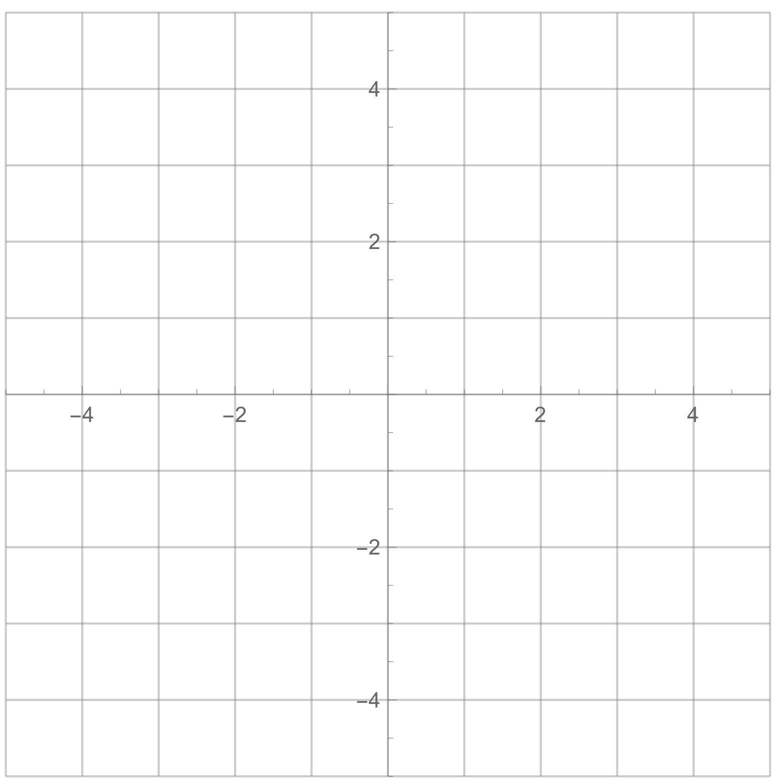
(a)

$$2x - y = 3$$



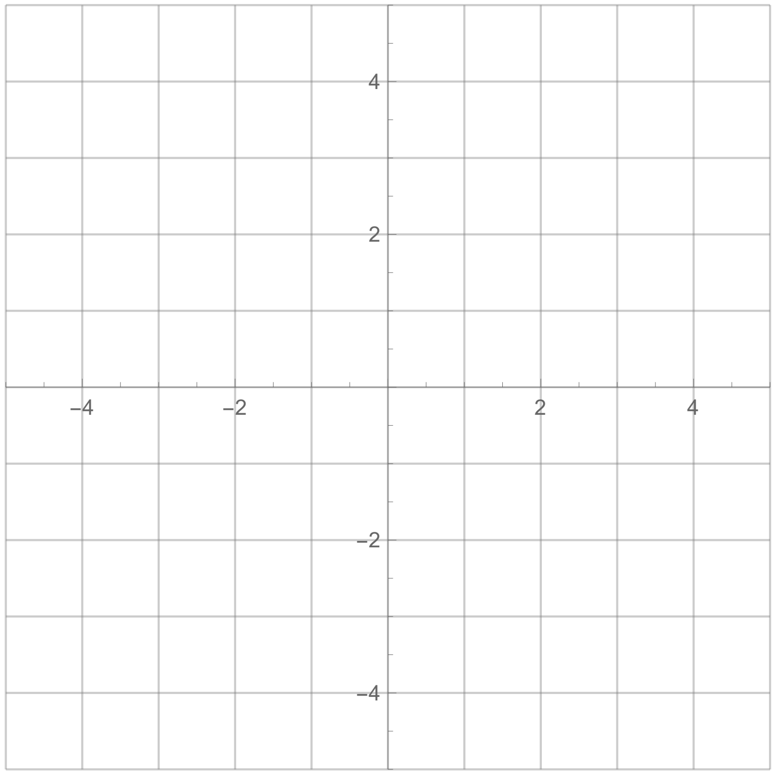
(b)

$$2x + 4y = 8$$



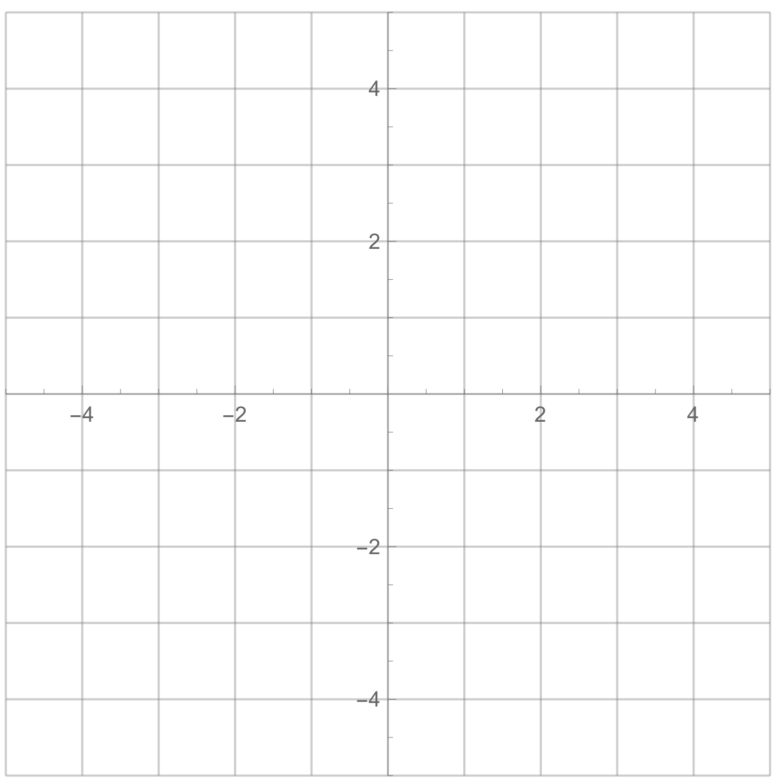
(c)

$$3x - 2y = 0$$



(d)

$$6x = 18$$



Section 2.5 - Exponential Functions

Definition 3 (Exponential Function)**.** *An* exponential function *is a function of the form*

$$f(x) = b^x, \ b > 0, \ b \neq 1.$$

b is called the base.

Why the restrictions on b?

- If $b = 1$, then $f(x) = 1^x = 1$ for all x values. Not a very interesting function!
- As an example of the case when $b < 0$, suppose $b = -1$. Then

  $$f\left(\frac{1}{2}\right) = (-1)^{1/2} = \sqrt{-1} = i$$

  an imaginary number! This kind of thing will always happen if b is negative.
- If $b = 0$, then for negative x values, f is not defined. For example,

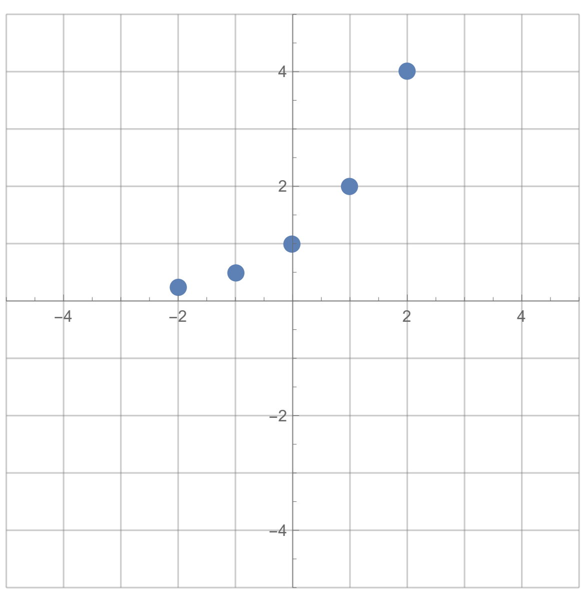
  $$f(-1) = 0^{-1} = \frac{1}{0} = undefined.$$

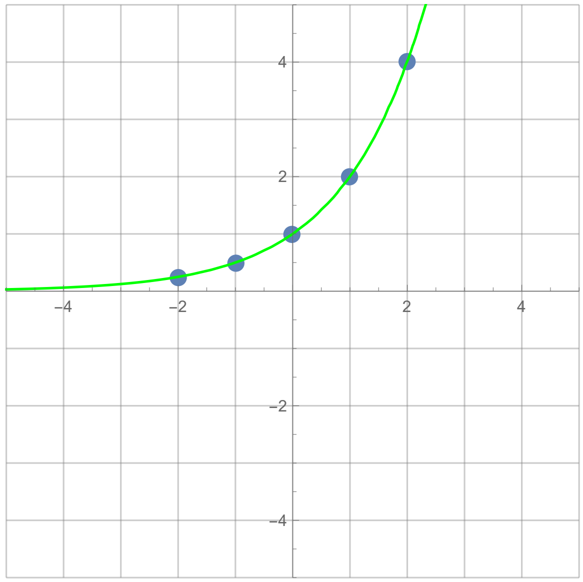
Let's get an idea of what these functions look like by graphing a few of them.

Example 4. *Sketch the graph of $f(x) = 2^x$.*

Solution. *Let's just plug in a few test points and connect the dots.*

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4



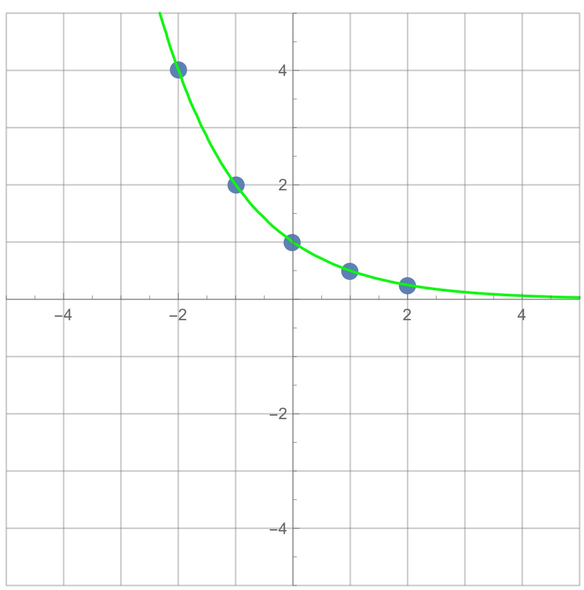


When $b > 1$, the graph of $f(x) = b^x$ has the same basic shape as 2^x, but may be steeper or more gradual. Let's see what happens when $b < 1$.

Example 5. *Sketch the graph of* $f(x) = \left(\frac{1}{2}\right)^x$.

Solution. *Let's just plug in a few test points and connect the dots.*

x	-2	-1	0	1	2
$f(x)$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



Notice that

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$$

so that when $b < 1$, we can set $b = \frac{1}{c}$ and have $c > 1$ and

$$f(x) = b^x = \left(\frac{1}{c}\right)^x = c^{-x}.$$

So, we can always keep the base larger than 1 by using a minus sign in the exponent if necessary.

Properties of Exponential Functions.

Property 1 (Graphical Properties of Exponential Functions)**.** *The graph of $f(x) = b^x$, $b > 0$, $b \neq 1$ satisfies the following properties:*

(1) *All graphs pass through the point $(0, 1)$.*

(2) *All graphs are continuous.*

(3) *The x-axis is a horizontal asymptote.*

(4) *b^x is increasing if $b > 1$.*

(5) *b^x is decreasing if $0 < b < 1$.*

Property 2 (General Properties of Exponents)**.** *Let $a, b > 0$, $a, b \neq 1$, and x, y be real numbers. The following properties are satisfied:*

(1) $a^x a^y = a^{x+y}$, $\dfrac{a^x}{a^y} = a^{x-y}$, $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$, $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

(2) $a^x = a^y$ *if and only if* $x = y$

(3) $a^x = b^x$ *for all* x *if and only if* $a = b$

A Special Number: e. There is one number that occurs in applications a lot: the natural number e. One definition of e is the value which the quantity

$$\left(1 + \frac{1}{x}\right)^x$$

approaches as x tends towards ∞.

This number often shows up in growth and decay models, such as population growth, radioactive decay, and continuously compounded interest. If c is the initial amount of the measured quantity, and r is the growth/decay rate of the quantity ($r > 0$ is for growth, $r < 0$ is for decay), then the amount after time t is given by

$$A = ce^{rt}.$$

Example 6. *In 2013, the estimated world population was* 7.1 *billion people with a relative growth rate of* 1.1%*.*

(a) *Write a function modeling the world population t years after 2013.*

(b) *What is the expected population in 2015? 2025? 2035?*

Solution.

(a) *We will write our function to output in billions. We will treat $t = 0$ as the year 2013, so that the initial population for this model is $c = 7.1$. The relative growth rate is 1.1%, which we must convert to a decimal before using $r = 0.011$. t will measure the years since 2013. Plugging these into the model, we get*

$$\textit{Population} = P = 7.1e^{0.011t} \textit{ billion.}$$

(b) *To find the estimated population in these years, we just need to plug the appropriate t-value into the model above.*

2015) $t = 2$

$$P = 7.1e^{0.011(2)} = 7.1e^{0.022} \approx 7.26 \textit{ billion}$$

2025) $t = 12$

$$P = 7.1e^{0.011(12)} = 7.1e^{0.132} \approx 8.1 \textit{ billion}$$

2035) $t = 22$

$$P = 7.1e^{0.011(22)} = 7.1e^{0.242} \approx 9.04 \textit{ billion}$$

Section 2.6 - Logarithmic Functions

Before we can accurately talk about what logarithms are, let's first remind ourselves about inverse functions.

Inverse Functions. The inverse of a function is given by running the function backwards. But when can we do this?

Consider the function $f(x) = x^2$. If we run f backwards on the value 1, what x-value do we get?

Since $(1)^2 = 1$ and $(-1)^2 = 1$, we get *two* values when we run x^2 backward! So x^2 is not invertible.

This shows that not every function is invertible. To get the inverse of a function, we need each range value to come from *exactly one* domain value. We call such functions *one-to-one*.

If we have a one-to-one function

$$y = f(x)$$

we can form the *inverse function* by switching x and y and solving for y:

$$x = f(y) \xrightarrow{\text{solve for } y} y = f^{-1}(x).$$

Logarithms. We will focus on one particular inverse function: the inverse of the function $f(x) = b^x$ ($b > 0$, $b \neq 1$).

Definition 4 (Logarithm)**.** *The* logarithm of base b *is defined as the inverse of* b^x*. That is,*

$$y = b^x \iff x = \log_b y.$$

Since the domain and range switch when we take inverses, we have

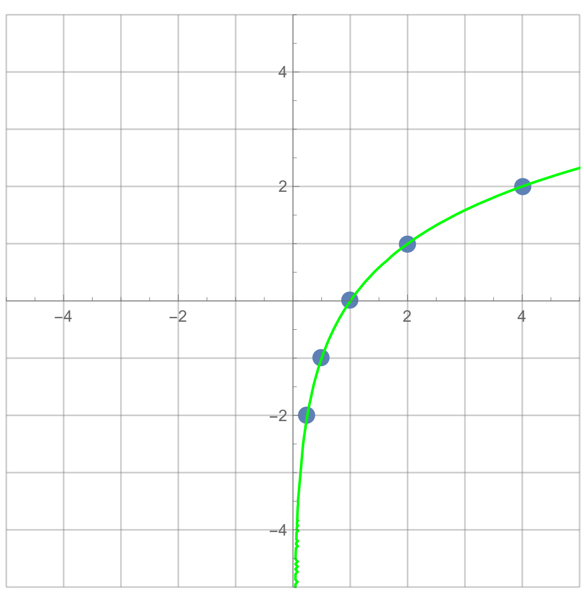
function	domain	range
$f(x) = b^x$	$(-\infty, \infty)$	$(0, \infty)$
$f(x) = \log_b x$	$(0, \infty)$	$(-\infty, \infty)$

Let's look at one example of a graph of a logarithmic function.

Example 7. *Sketch the graph of* $f(x) = \log_2 x$*.*

Solution. *To get the points for this, we can just recognize that it is the inverse of* 2^x *so we take each of those points and flip the* x *and* y *coordinates. This gives*

x	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
$f(x)$	-2	-1	0	1	2



Properties of Logarithms. Since logarithms are inverse to exponential functions, we get some convenient properties for logarithms:

Property 3 (Properties of Logarithms)**.** *Let $b, M, N > 0$, $b \neq 1$, and p, x be real numbers. Then*

(1) $\log_b 1 = 0$

(2) $\log_b b = 1$

(3) $\log_b b^x = x$

(4) $b^{\log_b x} = x$

(5) $\log_b MN = \log_b M + \log_b N$

(6) $\log_b \dfrac{M}{N} = \log_b M - \log_b N$

(7) $\log_b M^p = p \log_b M$

(8) $\log_b M = \log_b N$ *if and only if* $M = N$

Properties 3, 4, and 7 above are incredibly important to us as we will use them frequently in the study of financial mathematics! Learn these properties well!!

The Natural Logarithm. Just as with exponential functions, if we choose our base to be the number e, we get a special logarithm, the *natural logarithm.*

$$\log_e x = \ln x.$$

We can actually rewrite a logarithm in any base in terms of ln:

$$\log_b x = \frac{\ln x}{\ln b}$$

(See the textbook for a proof of this.)

Recall that exponential growth/decay models are of the form

$$A = ce^{rt}.$$

Using the natural logarithm, we can solve for the rate of growth/decay, r, and the time elapsed, t. Let's see this in an example.

Example 8. *The isotope carbon-*14 *has a half-life (the time it takes for the isotope to decay to half of its original mass) of* 5730 *years.*

(a) *At what rate does carbon-*14 *decay?*

(b) *How long would it take for* 90% *of a chunk of carbon-*14 *to decay?*

Solution.

(a) *Suppose we have an initial mass of* M_0. *After half of it decays, the mass will be* $\frac{M_0}{2}$ *and this happens after* $t = 5730$ *years has elapsed. Plugging all this into our model, we get*

$$\frac{M_0}{2} = M_0 e^{r(5730)} \Longleftrightarrow \frac{1}{2} = e^{5730r}$$

Applying the natural log to each side gives

$$\ln \frac{1}{2} = \ln e^{5730r}$$

Using properties of logarithms, we have

$$\ln \frac{1}{2} = \ln 2^{-1} = -\ln 2$$

and

$$\ln e^{5730r} = 5730r \ln e = 5730r$$

so that

$$-\ln 2 = 5730r.$$

Solving for r, *we get*

$$r = -\frac{\ln 2}{5730} \approx -0.00012$$

This means that carbon-14 decays at a rate of 0.12% *per year.*

(b) *If the mass of* M_0 *loses* 90% *of its mass, we're looking for the time it takes for only* $0.1M_0$ *to remain. So,*

$$0.1M_0 = M_0 e^{-0.00012t}$$

and canceling the M_0*'s gives*

$$0.1 = e^{-0.00012t}.$$

Hit both sides of this with ln *to get*

$$\ln 0.1 = \ln e^{-0.00012t} = -0.00012t.$$

Solve for t

$$t = -\frac{\ln 0.1}{0.00012} \approx 19,188.21.$$

So, it would take about 12,188.21 *years for* 90% *of the original mass to decay.*