Finite Math - J-term 2017

Lecture Notes - 1/19/2017

Homework

- Section 4.5 - 77, 78, 79, 80
- Section 4.6 - 9, 12, 14, 15, 17, 18, 21, 26, 29, 32, 37, 39, 41, 42, 45, 46, 55
- Section 5.1 - 9, 11, 12, 13, 14, 17, 29, 30

Section 4.5 - Inverse of a Square Matrix

Cryptography. Suppose we represent letters by numbers as follows

Blank	0	I	9	R	18
A	1	J	10	S	19
B	2	K	11	T	20
C	3	L	12	U	21
D	4	M	13	V	22
E	5	N	14	W	23
F	6	O	15	X	24
G	7	P	16	Y	25
H	8	Q	17	Z	26

Then, for example, the message "SECRET CODE" would correspond to the sequence

$$19\ \ 5\ \ 3\ \ 18\ \ 5\ \ 20\ \ 0\ \ 3\ \ 15\ \ 4\ \ 5$$

The goal of Cryptography is to encode messages in a different sequence which can only be translated back to the message using a decoder.

Definition 1 (Encoding matrix/Decoding matrix)**.** *Any matrix with positive integer elements whose inverse exists can be used as an* encoding matrix*. The inverse of an encoding matrix is a* decoding matrix*.*

To encode a message, we must first decide on a encoding matrix A. If A is a $n \times n$ matrix, then we create another matrix $n \times p$ matrix B by entering the message going down columns and taking as many columns as necessary to fit the whole message. Note that the number of rows of B MUST MATCH the size of A. If there are extra entries in B after fitting the whole message, just fill them with 0's.

Example 1. *Encode the message "SECRET CODE" using the encoding matrix*

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

Solution. *We first make the matrix B*

$$B = \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{bmatrix}$$

Then to encode the message we find the product

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 19 & 3 & 5 & 0 & 15 & 5 \\ 5 & 18 & 20 & 3 & 4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 38+15 & 6+54 & 10+60 & 0+9 & 30+12 & 10+0 \\ 19+5 & 3+18 & 5+20 & 0+3 & 15+4 & 5+0 \end{bmatrix} \\ &= \begin{bmatrix} 53 & 60 & 70 & 9 & 42 & 10 \\ 24 & 21 & 25 & 3 & 19 & 5 \end{bmatrix} \end{aligned}$$

So the coded message is

$$53 \;\; 24 \;\; 60 \;\; 21 \;\; 70 \;\; 25 \;\; 9 \;\; 3 \;\; 42 \;\; 19 \;\; 10 \;\; 5$$

Example 2. *A message was encoded with A from the previous example. Decode the sequence*

$$29 \;\; 12 \;\; 69 \;\; 28 \;\; 70 \;\; 25 \;\; 111 \;\; 43$$

Solution. *First we have to invert the encoding matrix to get the decoding matrix*

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

Make a matrix out of the coded message in the same way as above

$$C = \begin{bmatrix} 29 & 69 & 70 & 111 \\ 12 & 28 & 25 & 43 \end{bmatrix}$$

and find the product

$$\begin{aligned} A^{-1}C &= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 29 & 69 & 70 & 111 \\ 12 & 28 & 25 & 43 \end{bmatrix} \\ &= \begin{bmatrix} -29+36 & -69+84 & -70+75 & -111+129 \\ 29-24 & 69-56 & 70-50 & 111-86 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 15 & 5 & 18 \\ 5 & 13 & 20 & 25 \end{bmatrix} \end{aligned}$$

Example 3. *Use the encoding matrix*

$$E = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}.$$

(a) *Encode the message "MATH IS FUN" using E.*

(b) *Decode the sequence*

39 60 91 65 110 125 6 7 16 44 63 113 37 53 87

Section 4.6 - Matrix Equations and Systems of Linear Equations

Matrix Equations.

Theorem 1. *Assume that all products and sums are defined for the indicated matrices A, B, C, I, and 0 (where 0 stands for the zero matrix). Then*

- *Addition Properties*
  (1) *Associative*
  $$(A+B)+C = A+(B+C)$$
  (2) *Commutative*
  $$A+B = B+A$$
  (3) *Additive Identity*
  $$A+0 = 0+A = A$$
  (4) *Additive Inverse*
  $$A+(-A) = (-A)+A = 0$$
- *Multiplication Properties*
  (1) *Associative Property*
  $$A(BC) = (AB)C$$
  (2) *Multiplicative Identity*
  $$AI = IA = A$$
  (3) *Multiplicative Inverse*
  If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$
- *Combined Properties*
  (1) *Left Distributive*
  $$A(B+C) = AB+AC$$
  (2) *Right Distributive*
  $$(B+C)A = BA+CA$$

- *Equality*

  (1) *Addition*
  If $A = B$*, then* $A + C = B + C$

  (2) *Left Multiplication*
  If $A = B$*, then* $CA = CB$

  (3) *Right Multiplication*
  If $A = B$*, then* $AC = BC$

We can use the rules above to solve various matrix equations. In the next 3 examples, we will assume all necessary inverses exists.

Example 4. *Suppose* A *is an* $n\times n$ *matrix and* B *and* X *are* $n\times 1$ *column matrices. Solve the matrix equation for* X

$$AX = B.$$

Solution. *If we multiply both sides of this equation ON THE LEFT by* A^{-1} *we find*

$$A^{-1}(AX) = A^{-1}B \quad \Longrightarrow \quad (A^{-1}A)X = IX = X = A^{-1}B$$

Example 5. *Suppose* A *is an* $n \times n$ *matrix and* B*,* C*, and* X *are* $n \times 1$ *matrices. Solve the matrix equation for* X

$$AX + C = B.$$

Solution. *Begin by subtracting* C *to the other side*

$$AX + C = B \quad \Longrightarrow \quad AX = B - C$$

and now multiply on the left by A^{-1}

$$A^{-1}(AX) = A^{-1}(B-C) \quad \Longrightarrow \quad (A^{-1}A)X = IX = X = A^{-1}(B-C) = A^{-1}B-A^{-1}C$$

Example 6. *Suppose* A *and* B *are* $n\times n$ *matrices and* C *is an* $n\times 1$ *matrix. Solve the matrix equation for* X

$$AX - BX = C.$$

What size matrix is X*?*

Matrix Equations and Systems of Linear Equations. We can also solve systems of equations using the above ideas. These apply in the case that the system has the same number of variables as equations and the coefficient matrix of the system is invertible. If that is the case, for the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & + & \vdots & + & \ddots & + & a_{1n}x_n & = & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

We can create the matrix equation

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{1n} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then, if A is invertible (as is the case when the system is consistent and independent, i.e., exactly one solution), we have

$$X = A^{-1}B.$$

Example 7. *Solve the system of equations using matrix methods*

$$\begin{array}{ccccc} x & + & 2y & = & 1 \\ x & + & 3y & = & 3 \end{array}$$

Solution. *Begin by writing this system as a matrix equation*

$$AX = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = B$$

Our goal is to find $A^{-1}B$, so first find A^{-1}:

$$A^{-1} = \frac{1}{3-2} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

So we have $x = -3$ and $y = 2$.

Example 8. *Solve the system of equations using matrix methods*

$$\begin{array}{ccccc} 2x & + & y & = & 1 \\ 5x & + & 3y & = & -1 \end{array}$$

Solution. $x = 4$, $y = -7$

Section 5.1 - Linear Inequalities in Two Variables

Graphing Linear Inequalities in Two Variables. There are 4 types of linear inequalities

$$Ax + By \geq C \qquad\qquad Ax + By > C$$

$$Ax + By \leq C \qquad\qquad Ax + By < C$$

There is a simple procedure to graphing any of these. If equality is not allowed in an inequality, we call it a *strict inequality*, otherwise we simply call it an inequality.

Procedure.

(1) *Graph the line* $Ax + By = C$ *as a dashed line if the inequality is strict. Otherwise, graph it as a solid line.*

(2) *Choose a test point anywhere in the plane, as long as it is not the line. (The origin,* $(0, 0)$ *is often an easy choice here, but if it is on the line,* $(1, 0)$ *or* $(0, 1)$ *are also easy points to check.)*

(3) *Plug the point from step* (2) *into the inequality. Is the inequality true? Shade in the side of the line with that point. If the inequality is false, shade in the other side.*

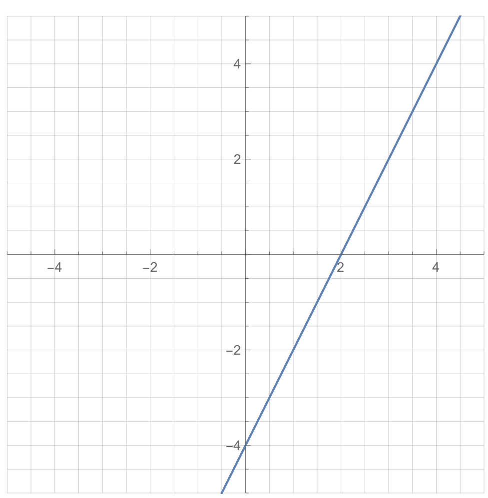
Example 9. *Graph the inequality*

$$6x - 3y \geq 12$$

Solution. *The line we want to graph is*

$$6x - 3y = 12 \quad \text{or} \quad y = 2x - 4.$$

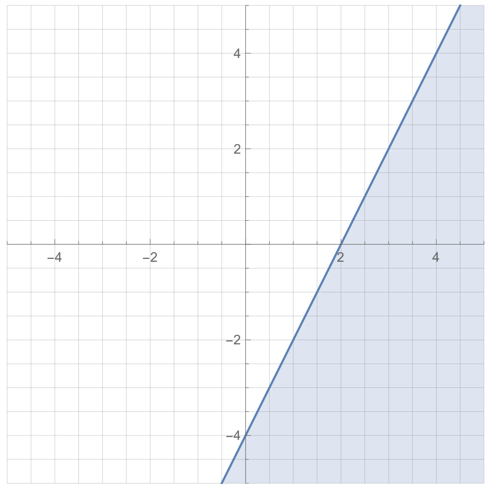
Since the inequality is not strict, we graph it with a solid line.



The point $(0,0)$ is not on the line, so we check that point in the inequality

$$6(0) - 3(0) = 0 \geq 12$$

This is false, so we shade in the side of the line without the origin.



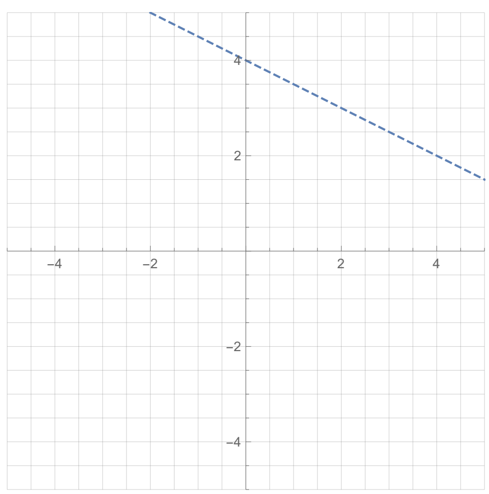
Example 10. *Graph the inequality*

$$4x + 8y < 32$$

Solution. *The line we want to graph is*

$$4x + 8y = 32 \quad or \quad y = -\frac{1}{2}x + 4.$$

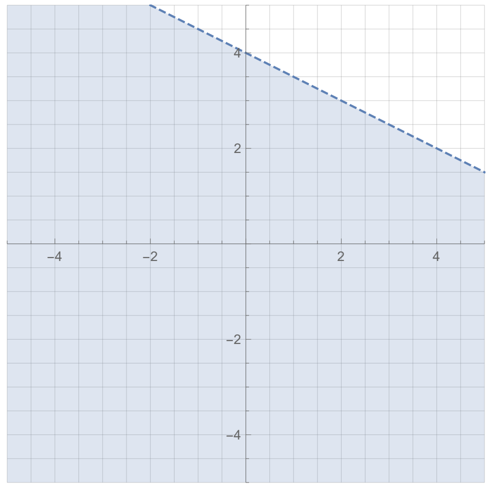
Since the inequality is strict, we graph it with a dashed line.



The point $(0,0)$ is not on the line, so we check that point in the inequality

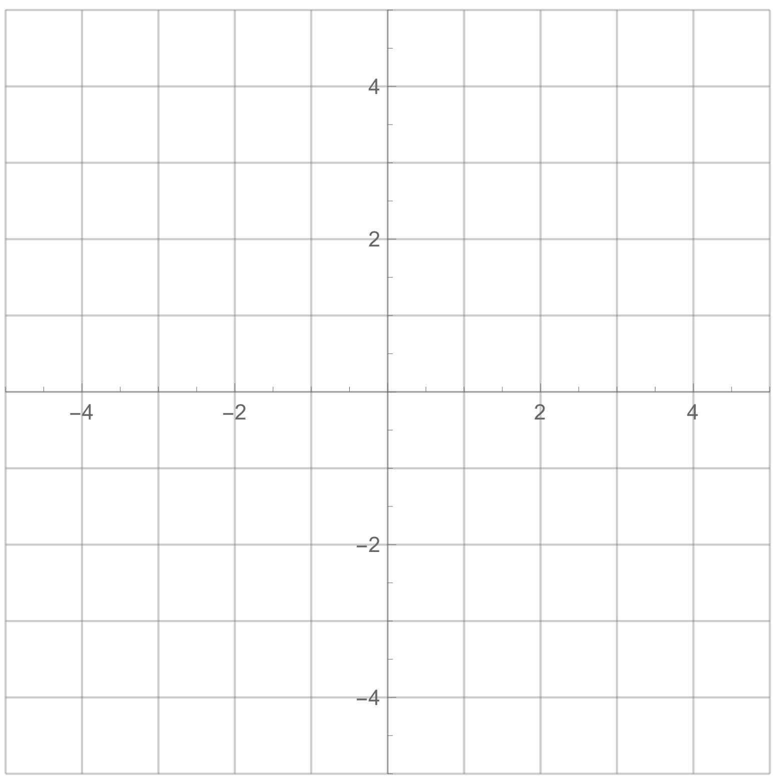
$$4(0) + 8(0) = 0 < 32$$

This is true, so we shade in the side of the line with the origin.



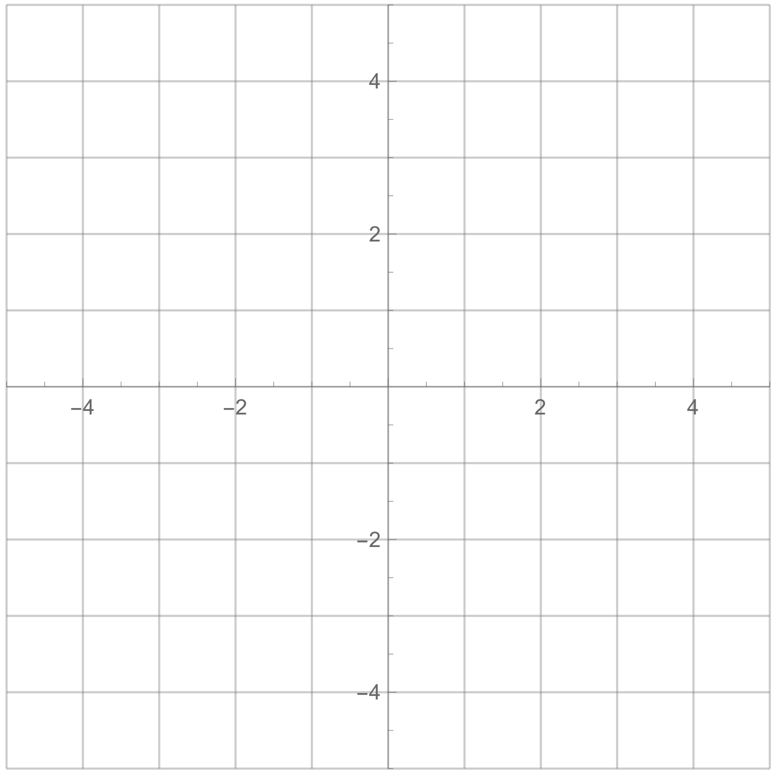
Example 11. *Graph the inequality*

$$2y \leq 10$$

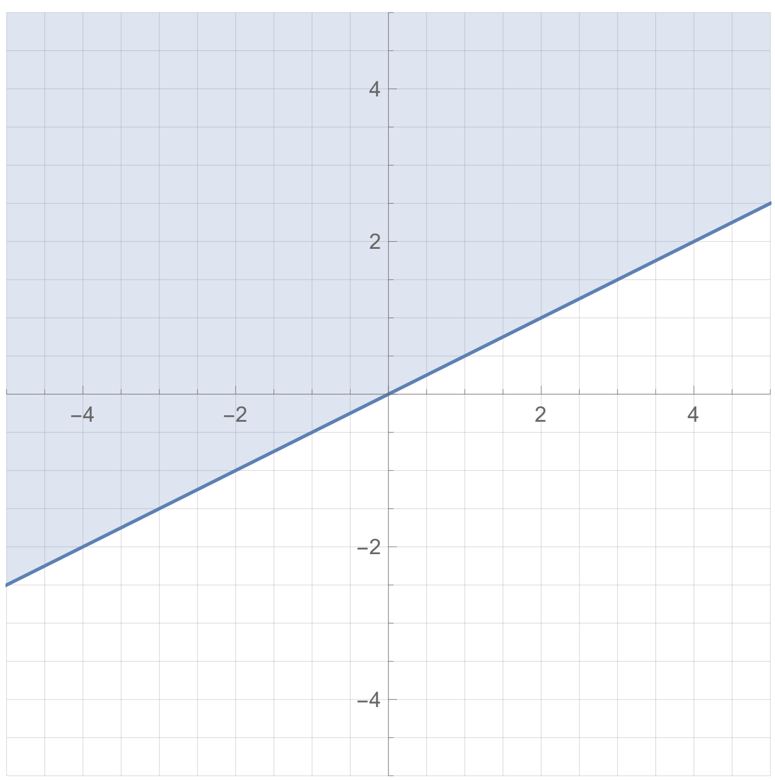


Example 12. *Graph the inequality*

$$2x - 5y > 10$$



Example 13. *Consider the graphed region below.*



(a) *Find an equation for the boundary of the region in the form* $Ax + By = C$.

(b) *Find a linear equality which describes this region.*

Solution.

(a) *Observing the graph, we see that the boundary line passes through* $(0, 0)$ *and* $(2, 1)$. *Using the point-slope form, we get*

$$y - 0 = \frac{1-0}{2-0}(x - 0)$$

which simplifies to

$$y = \frac{1}{2}x$$

Putting this in the required form gives

$$x - 2y = 0.$$

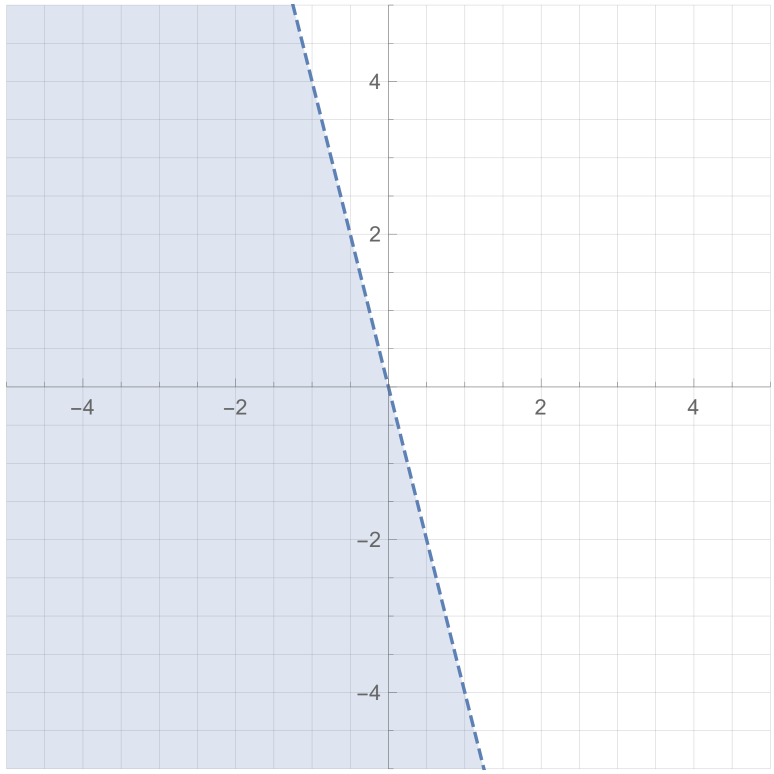
(b) *Because the boundary line is solid, we are going to replace* $=$ *with either* $\geq$ *or* $\leq$. *To figure out which one, we pick a test point which is not on the line and choose the inequality appropriately. If the test point comes from the shaded region, then we pick the inequality which makes the statement true. If the test point comes from outside the shaded region, pick the inequality which makes the statement false.*
Since $(0, 0)$ *actually is on this line, we will pick* $(1, 0)$ *as our test point. Notice that* $(1, 0)$ *is* outside *the shaded region. Plugging* $(1, 0)$ *into the equation gives*

$$x - 2y = 1 - 2(0) = 1 \boxed{?} 0.$$

Since $(1, 0)$ *is not in the shaded region, we need to pick the one of* $\geq$ *and* $\leq$ *to replace* $\boxed{?}$ *which makes the statement false. This means we choose* $\leq$ *giving that the inequality for this picture is*

$$x - 2y \leq 0.$$

Example 14. *Consider the graphed region below.*



(a) *Find an equation for the boundary of the region in the form* $Ax + By = C$.

(b) *Find a linear equality which describes this region.*