Compound Interest

Finite Math

9 January 2019

Average Daily Balance

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Example

A credit card has an annual interest rate of 19.99% *and interest is calculated using the average daily balance method. If the starting balance of a* 30*-day billing cycle is* \$523.18 *and purchases of* \$147.98 *and* \$36.27 *are posted on days* 12 *and* 25*, respectively, and a payment of* \$200 *is credited on day* 17*, what will be the balance on the card at the start of the next billing cycle?*

Now You Try It!

Example

A credit card has an annual interest rate of 19.99% *and interest is calculated using the average daily balance method. If the starting balance of a* 28*-day billing cycle is* \$696.21 *and purchases of* \$25.59, \$19.95, *and* \$97.26 *are posted on days* 6, 13, *and* 25, *respectively, and a payment of* \$140 *is credited on day* 8, *what will be the balance on the card at the start of the next billing cycle?*

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Solution

\$708.92

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Example

Suppose $\$5,000$ *is invested at* 12%*, compounded quarterly. How much is the investment worth after* 1 *year?*

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Definition (Compound Interest)

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- m = *number of compounding periods per year*
- i = *rate per compounding period*
- n = *total number of compounding periods*

Compound Interest

Alternately, one can reinterpret this formula as a function of time as

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

where A, P, r, and m have the same meanings as above and t is the time in years.

Compound Interest

Example

If $\$1,000$ is invested at 6% interest compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, what is the value of the investment after 8 years? Round answers to the nearest cent.

Now You Try It!

Example

If $\$2,000$ *is invested at* 7% *interest compounded (a) annually, (b) quarterly, (c) monthly, what is the amount after* 5 *years? How much interest is accrued in each case? Round answers to the nearest cent.*

Now You Try It!

Example

If $\$2,000$ is invested at 7% interest compounded (a) annually, (b) quarterly, (c) monthly, what is the amount after 5 years? How much interest is accrued in each case? Round answers to the nearest cent.

Solution

(a) $\$2805.10$ *with* $\$805.10$ *in interest.*

(b) $\$2829.56$ *with* $\$829.56$ *in interest.*

(c) $\$2835.25$ *with* $\$835.25$ *in interest.*

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$$\begin{aligned} A &= P\left(1 + \frac{r}{m}\right)^{mt} = P\left(1 + \frac{r}{m}\right)^{mt\cdot\frac{r}{r}} \\ &= P\left(1 + \frac{r}{m}\right)^{\left(\frac{m}{r}\right)rt} \end{aligned}$$

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$$A = Pe^{rt}.$$

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Principal P invested at an annual nominal rate r will have future value

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after time t (in years).

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Compounding interest continuously gives the absolute largest amount of interest that can be accumulated in the time period t.

Continuous Compound Interest

Example

If $\$1,000$ is invested at 6% interest compounded continuously, what is the value of the investment after 8 years? Round answers to the nearest cent.

Now You Try It!

Example

If $\$2,000$ *is invested at* 7% *compounded (a) annually, (b) quarterly, (c) monthly, (d) daily, (e) continuously, what is the amount after* 5 *years? Round answers to the nearest cent. (Assume* 365 *days in a year.)*

Now You Try It!

Example

If $\$2,000$ is invested at 7% compounded (a) annually, (b) quarterly, (c) monthly, (d) daily, (e) continuously, what is the amount after 5 years? Round answers to the nearest cent. (Assume 365 days in a year.)

Solution

(a) \$2805.10

(b) \$2829.56

(c) \$2835.25

(d) \$2838.04

(e) \$2838.14

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We can also look to see how long something will take to mature given the principal, the growth rate, and the desired future value. The power rule for logarithms comes especially in handy here: $\log_b M^p = p \log_b M$.

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Example

How long will it take $\$10,000$ *to grow to* $\$25,000$ *if it is invested at* 8% *compounded quarterly?*

Now You Try It!

Example

How long will it take money to triple if it is invested at (a) 5% compounded daily? (b) 6% compounded continuously? (round to 3 decimal places)

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How long will it take money to triple if it is invested at (a) 5% *compounded daily? (b)* 6% *compounded continuously? (round to* 3 *decimal places)*

Solution

(a) $8,021$ *days (about* 21.975 *years)*

(b) 18.310 *years*

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Example

The Russell Index tracks the average performance of various groups of stocks. On average, a $\$10,000$ investment in mid-cap growth funds over a 10-year period would have grown to $\$63,000$. What annual nominal rate would produce the same growth if interest were compounded (a) annually, (b) continuously? Express answers as a percentage, rounded to three decimal places.

Now You Try It!

Example

A promissory note will pay $\$50,000$ *at maturity* 6 *years from now. If you pay* $\$28,000$ *for the note now, what rate would you earn if interest were compounded (a) quarterly, (b) continuously?*

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Example

A promissory note will pay $\$50,000$ *at maturity* 6 *years from now. If you pay* $\$28,000$ *for the note now, what rate would you earn if interest were compounded (a) quarterly, (b) continuously?*

Solution

(a) 9.78%

(b) 9.66%