Basics of Linear Programming

Finite Math

22 January 2019

Corner Points

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Example

Solve the following system of linear inequalities graphically and find the corner points:

$$\begin{array}{rcrcr} x & + & y & \leq & 10 \\ 5x & + & 3y & \geq & 15 \\ -2x & + & 3y & \leq & 15 \\ 2x & - & 5y & \leq & 6 \end{array}$$

Now You Try It!

Example

Solve the following system of linear inequalities graphically and find the corner points:

$$\begin{array}{rcrcl} 5x & + & y & \geq & 20 \\ x & + & y & \geq & 12 \\ x & + & 3y & \geq & 18 \\ & & x & \geq & 0 \\ & & y & \geq & 0 \end{array}$$

Bounded and Unbounded Regions

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Question

Which of the regions in the last 4 examples are bounded? Which are unbounded?

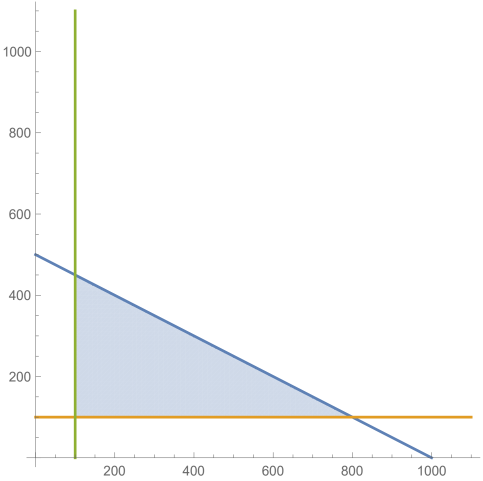
A Simple Linear Programming Problem

Example

A food vendor at a rock concert sells hot dogs for \$4 *each and hamburgers for* \$5 *each. She purchases hot dogs for* 50¢ *each and hamburgers for* \$1 *each. If she has* \$500 *to spend on supplies, and wants to bring at least* 100 *each of hot dogs and hamburgers, how many hot dogs and hamburgers should she buy to make the most money at the concert? (Assume she sells her entire inventory.) What is her maximum revenue?*

Solution

The feasible region of the problem is

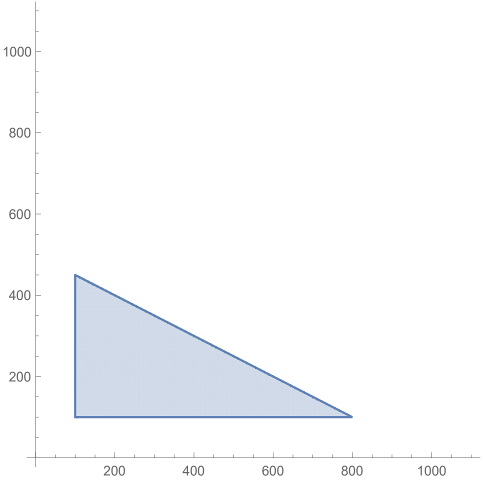


Solution

To figure out if she can make R_0 dollars in sales, we graph the line $4x + 5y = R_0$ and see if it hits the feasible region. If it does, it is possible to make that much in sales.

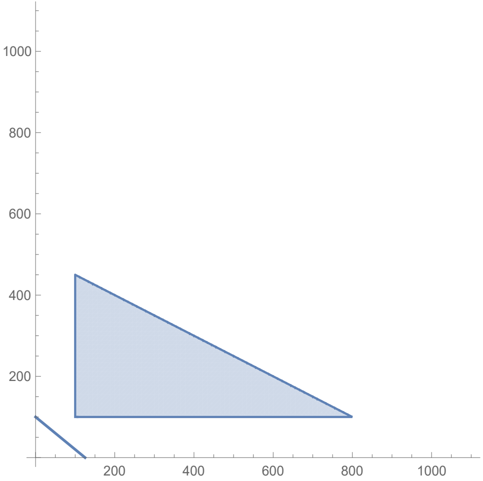
Solution

Here is the solution region again:



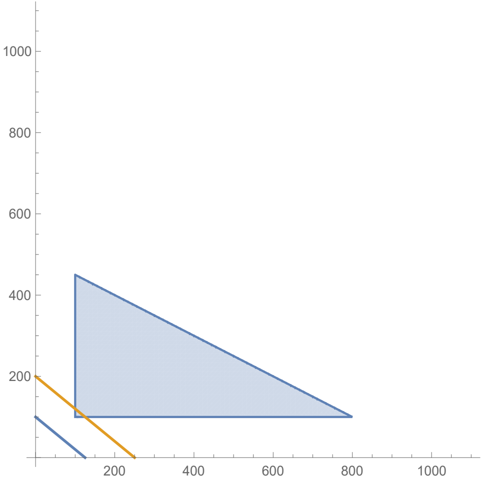
Solution

Add the revenue line for $R_0 = 500$.



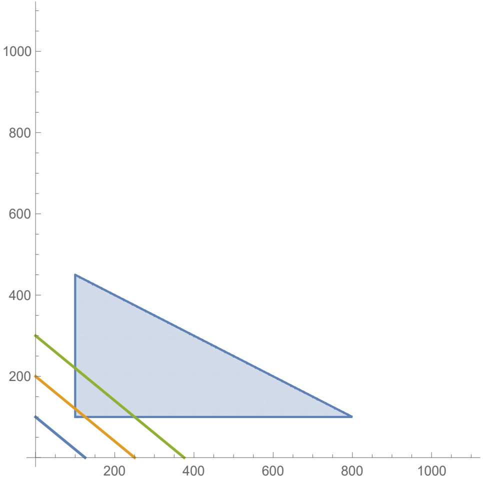
Solution

Add the revenue line for $R_0 = 1000$.



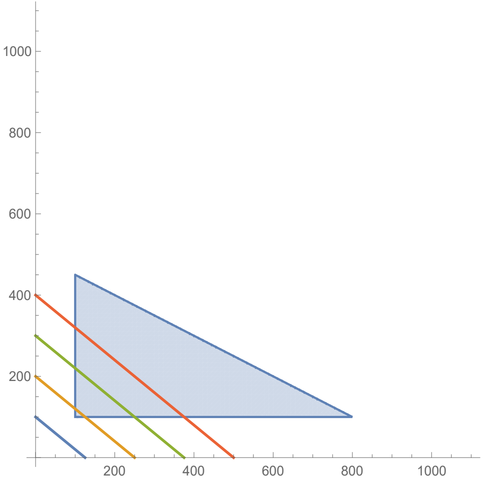
Solution

Add the revenue line for $R_0 = 1500$.



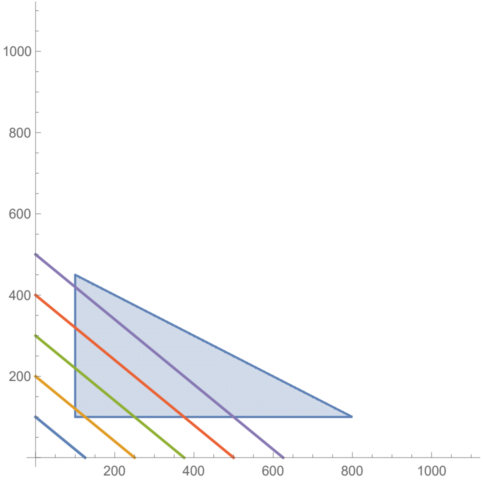
Solution

Add the revenue line for $R_0 = 2000$.



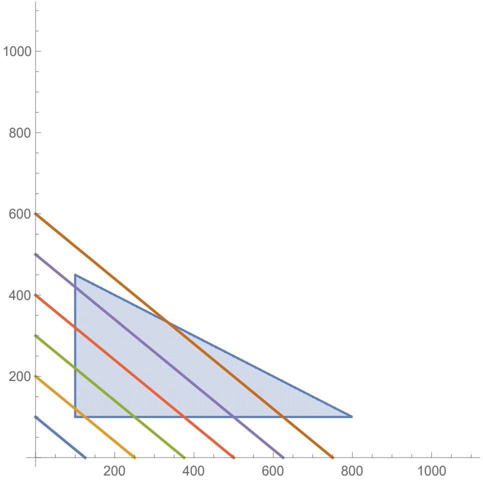
Solution

Add the revenue line for $R_0 = 2500$.



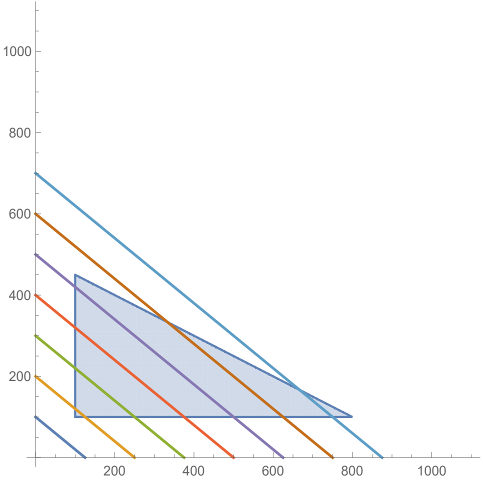
Solution

Add the revenue line for $R_0 = 3000$.



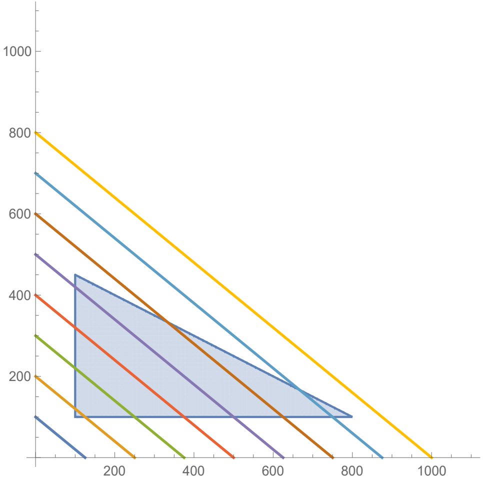
Solution

Add the revenue line for $R_0 = 3500$.



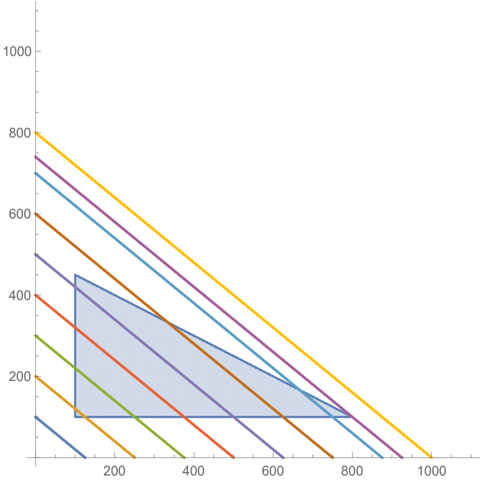
Solution

Add the revenue line for $R_0 = 4000$.



Solution

The line for max revenue is with $R_0 = 3700$.



General Description of Linear Programming

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In a *linear programming problem*, we are concerned with *optimizing* (finding the maximum and minimum values, called the *optimal values*) of a linear *objective function* z of the form

$$z = ax + by$$

where a and b are not both zero and the *decision variables* x and y are subject to *constraints* given by linear inequalities.

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where a and b are not both zero and the *decision variables* x and y are subject to *constraints* given by linear inequalities.Additionally, x and y must be nonnegative, i.e., $x \geq 0$ and $y \geq 0$.

When Can We Solve This?

Theorem (Fundamental Theorem of Linear Programming)

If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one or more of the corner points of the feasible region.

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(A) *If the feasible region for a linear programming problem is bounded, then both the maximum value and the minimum value of the objective function always exist.*

(B) *If the feasible region is unbounded and the coefficients of the objective function are positive, then the minimum value of the objective function exists, but the maximum value does not.*

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Theorem (Existence of Optimal Solutions)

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(B) *If the feasible region is unbounded and the coefficients of the objective function are positive, then the minimum value of the objective function exists, but the maximum value does not.*

(C) *If the feasible region is empty, then both the maximum value and the minimum value of the objective function do not exist.*

Geometric Method for Solving Linear Programming Problems

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2. *Construct a corner point table listing the value of the objective function at each corner point.*
3. *Determine the optimal solution(s) from the table in Step 2 (smallest=minimum, largest=maximum).*
4. *For an applied problem, interpret the optimal solution(s) in terms of the original problem.*

Linear Programming Example

Example

Maximize and minimize $z = 3x + y$ *subject to the inequalities*

$$\begin{aligned} 2x + y &\leq 20 \\ 10x + y &\geq 36 \\ 2x + 5y &\geq 36 \\ x, y &\geq 0 \end{aligned}$$

Now You Try It!

Example

Maximize and minimize $z = 2x + 3y$ *subject to*

$$\begin{aligned} 2x + y &\geq 10 \\ x + 2y &\geq 8 \\ x, y &\geq 0 \end{aligned}$$

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Example

Maximize and minimize $z = 2x + 3y$ *subject to*

$$\begin{aligned} 2x + y &\geq 10 \\ x + 2y &\geq 8 \\ x, y &\geq 0 \end{aligned}$$

Solution

Minimum of $z = 14$ *at* $(4, 2)$. *No maximum.*

Now You Try It!

Example

Maximize and minimize $P = 30x + 10y$ *subject to*

$$\begin{aligned} 2x + 2y &\geq 4 \\ 6x + 4y &\leq 36 \\ 2x + y &\leq 10 \\ x, y &\geq 0 \end{aligned}$$

Now You Try It!

Example

Maximize and minimize $P = 30x + 10y$ *subject to*

$$\begin{aligned} 2x + 2y &\geq 4 \\ 6x + 4y &\leq 36 \\ 2x + y &\leq 10 \\ x, y &\geq 0 \end{aligned}$$

Solution

Minimum of $P = 20$ *at* $(0, 2)$. *Maximum of* $P = 150$ *at* $(5, 0)$.

Now You Try It!

Example

Maximize and minimize $P = 3x + 5y$ *subject to*

$$\begin{aligned} x + 2y &\leq 6 \\ x + y &\leq 4 \\ 2x + 3y &\geq 12 \\ x, y &\geq 0 \end{aligned}$$

Now You Try It!

Example

Maximize and minimize $P = 3x + 5y$ *subject to*

$$\begin{aligned} x + 2y &\leq 6 \\ x + y &\leq 4 \\ 2x + 3y &\geq 12 \\ x, y &\geq 0 \end{aligned}$$

Solution

No optimal solutions.