Permutations and Combinations

Finite Math

28 January 2019

Factorial

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Definition (Factorial)

For a natural number n,

$$\begin{aligned} n! &= n(n-1)(n-2)\cdot\cdots\cdot 2\cdot 1 \\ 0! &= 1 \end{aligned}$$

Factorial

From this definition, we can see that

$$n! = n \cdot (n-1)! = n(n-1) \cdot (n-2)! = \cdots ,$$

that is, we can explicitly write out as many of the largest numbers as we need, then write the rest as a smaller factorial.

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$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!$$

if we wanted to bring special attention to 10 through 7.

Example

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Find

(a) $6!$

(b) $\dfrac{10!}{9!}$

(c) $\dfrac{10!}{7!}$

(d) $\dfrac{5!}{0!3!}$

(e) $\dfrac{20!}{3!17!}$

Now You Try It!

Example

Find

(a) $7!$

(b) $\dfrac{8!}{4!}$

(c) $\dfrac{8!}{4!(8-4)!}$

Permutations

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$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$$

possible arrangements, or permutations.

Permutations

Theorem (Permutations of n Objects)

The number of permutations of n distinct objects without repetition, denoted by $_nP_n$, is

$$_nP_n = n(n-1)\cdots 2\cdot 1 = n!.$$

Permutations of Subsets

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Definition (Permutation of n Objects Taken r at a Time)

A permutation of a set of n distinct objects taken r at a time without repetition is an arrangement of r of the n objects in a specific order.

Deriving a Formula for Permutations

If we have n things, and we want to create a permutation using r of them we have: n choices for the first slot, $n-1$ choices for the second, $n-2$ for the third, all the way up to $n-r+1$ options for the r^{th} slot.

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&= \frac{n!}{(n-r)!}
\end{aligned}$$

Permutations of Subsets

Theorem (Number of Permutations of n Objects Taken r at a Time)

The number of permutations of n distinct objects taken r at a time without repetition is given by

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!} $$

Example

Example

Given the set $\{A, B, C, D\}$, how many permutations are possible for this set of 4 objects taken 2 at a time?

Now You Try It!

Example

Find the number of permutations of 30 *objects taken* 4 *at a time.*

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Find the number of permutations of 30 *objects taken* 4 *at a time.*

Solution

$$_{30}P_4 = \frac{30!}{(30-4)!} = \frac{30!}{26!} = 30 \cdot 29 \cdot 28 \cdot 27 = 657,720$$

Combinations

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Definition (Combinations)

A combination of a set of n distinct objects taken r at a time without repetition is an r-element subset of the set of n objects. The arrangement of the elements in the subset does not matter.

Deriving a Formula for Combinations

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- Step 2: Put the r elements into a specific order. This is just a permutation of r elements, so there are $r!$ ways to do this.

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So, we can solve for ${}_nC_r$ to get

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}. $$

Combinations

Theorem (Number of Combinations of n Objects Taken r at a Time)

The number of combinations of n distinct objects taken r at a time without repetition is given by

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} $$

Example

Example

Form a committee of 12 people.

(a) *In how many ways can we choose a chairperson, a vice-chairperson, a secretary, and a treasurer, assuming that one person cannot hold more than one position?*

(b) *In how many ways can we choose a subcommittee of 4 people?*

Another Example

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Find the number of combinations of 30 *objects taken* 4 *at a time.*

Now You Try It!

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How many ways can a 3-person subcommittee be selected from a committee of 7 people? How many ways can a president, vice-president, and secretary be chosen from a committee of 7 people?

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How many ways can a 3-person subcommittee be selected from a committee of 7 people? How many ways can a president, vice-president, and secretary be chosen from a committee of 7 people?

Solution

35; 210

Now You Try It!

Example

Find the number of combinations of 67 *objects taken* 5 *at a time.*

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Find the number of combinations of 67 *objects taken* 5 *at a time.*

Solution

9,657,648

Poker Hands!

Example

Suppose we ave a standard 52-card deck and we are considering 5-card poker hands.

(a) *How many hands have 3 hearts and 2 spades?*

(b) *How many hands have all the same suit? (I.e., what is the number of different flushes?)*

(c) *How many possible pairs are there? (The other three cards have a different number from the pair and each other.)*

(d) *How many possible 3 of a kinds are there? (The other two cards have a different number from the 3 of a kind and from each other.)*

(e) *How many full houses are possible? (A full house consists of a three of a kind and a pair, each from a different number.)*