Exponential and Logarithmic Functions

Math 130 - Essentials of Calculus

16 September 2019

Exponential Functions

An *exponential function* is a function of the form

$$f(x) = b^x$$

where b is a positive constant called the *base*. The domain of an exponential function is $\mathbb{R} = (-\infty, \infty)$. A slightly more general exponential function is

$$f(x) = C \cdot b^x$$

where C represents the initial value of f (because $f(0) = C$).

Example

Graph the following exponential functions:

1. $f(x) = 2^x$
2. $f(x) = \left(\frac{1}{2}\right)^x$

Properties of Exponential Functions

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Simplify the following expressions:

(a) $x^5 x^3$ *(b)* $\dfrac{b^9}{b^3}$ *(c)* $(m^2 n)^4$ *(d)* $(2xy^2)^3$

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Under ideal conditions a certain bacteria population is known to double every 3 hours. Suppose that there are initially 100 bacteria.

1. *What is the size of the population after 15 hours?*
2. *What is the size of the population after t hours?*
3. *Estimate the size of the population after 20 hours.*

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The exponential function often shows up in "idealized" models of natural phenomena, such as assuming a population always grows at a specific rate. Another place it shows up is with continuously compounded interest (interest is compounded at every instant of time), in which case the formula for future value follows the form $A = Pe^{rt}$.

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The *inverse* of an exponential function is called a *logarithmic function*.

Definition

Let $b > 0$ be a real number with $b \neq 1$. The logarithm with base b is the function

$$f(x) = \log_b x$$

which satisfies the property

$$\log_b x = y \iff b^y = x.$$

When the base of the logarithm is the natural number e, we write $\ln x$ instead and call it the natural logarithm.

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Find the exact value of the following

1. $\log_2 64$

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5. $e^{\ln 4}$

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8. $\ln e^x$

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9. $e^{\ln x}$

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4. *Solve for x in the equation:* $e^{-x} = 5$