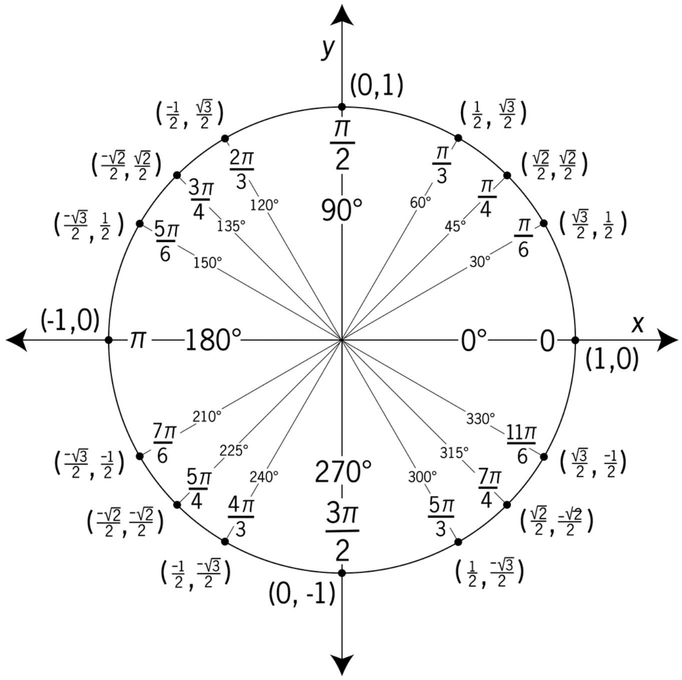
Trig Primer



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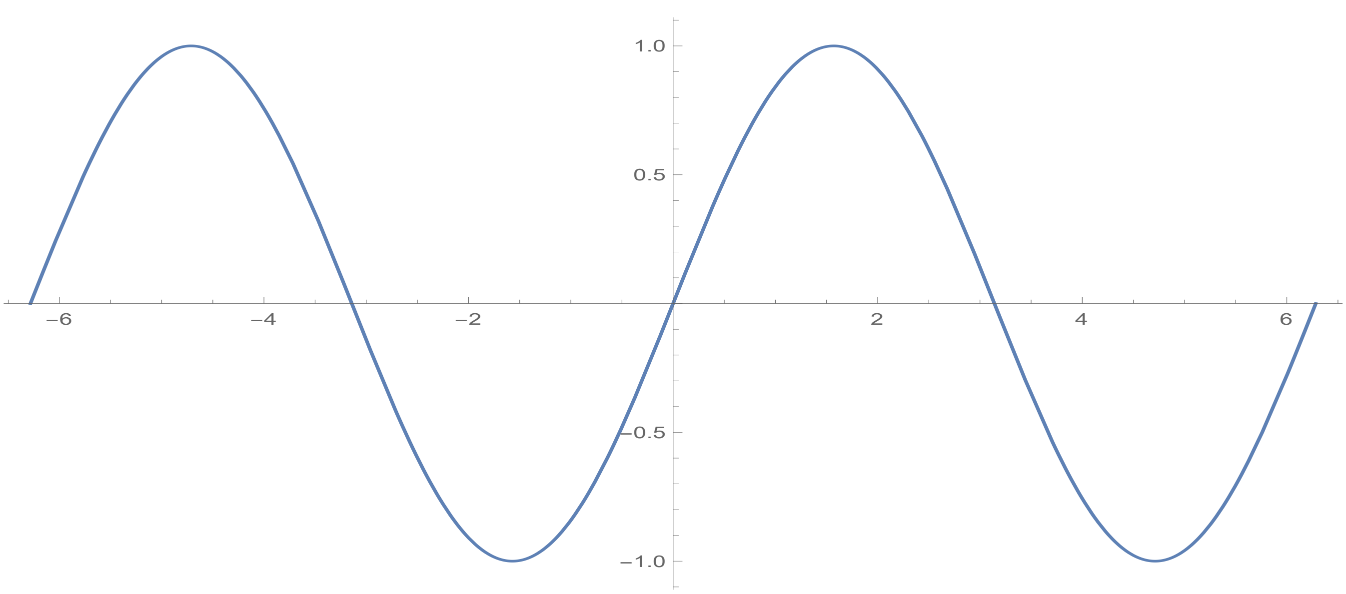
1. Graphs of Trig Functions

1.1. Sine. In the unit circle above, the y-values of all the coordinates correspond to the value of $\sin\theta$ at that angle. Summarizing a few values in a table, we have

$\theta =$	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta =$	0	1	0	-1	0	1	0	-1	0

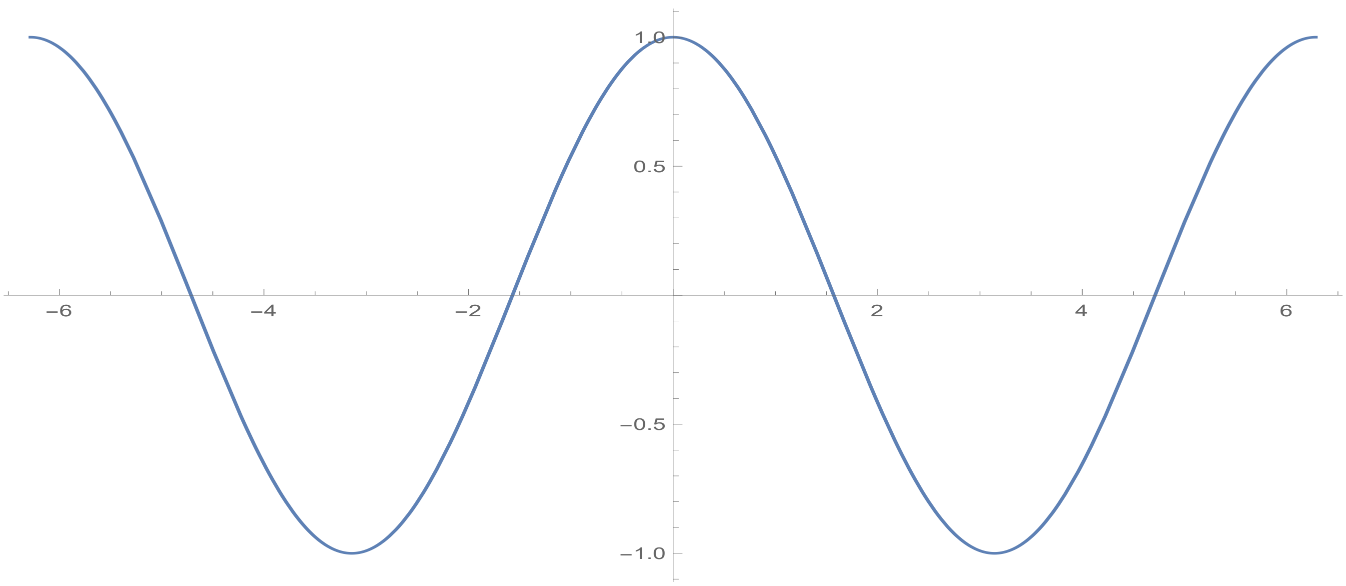
The graph of sine is then



1.2. **Cosine.** In the unit circle above, the x-values of all the coordinates correspond to the value of $\cos\theta$ at that angle. Summarizing a few values in a table, we have

$\theta =$	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\theta =$	1	0	-1	0	1	0	-1	0	1

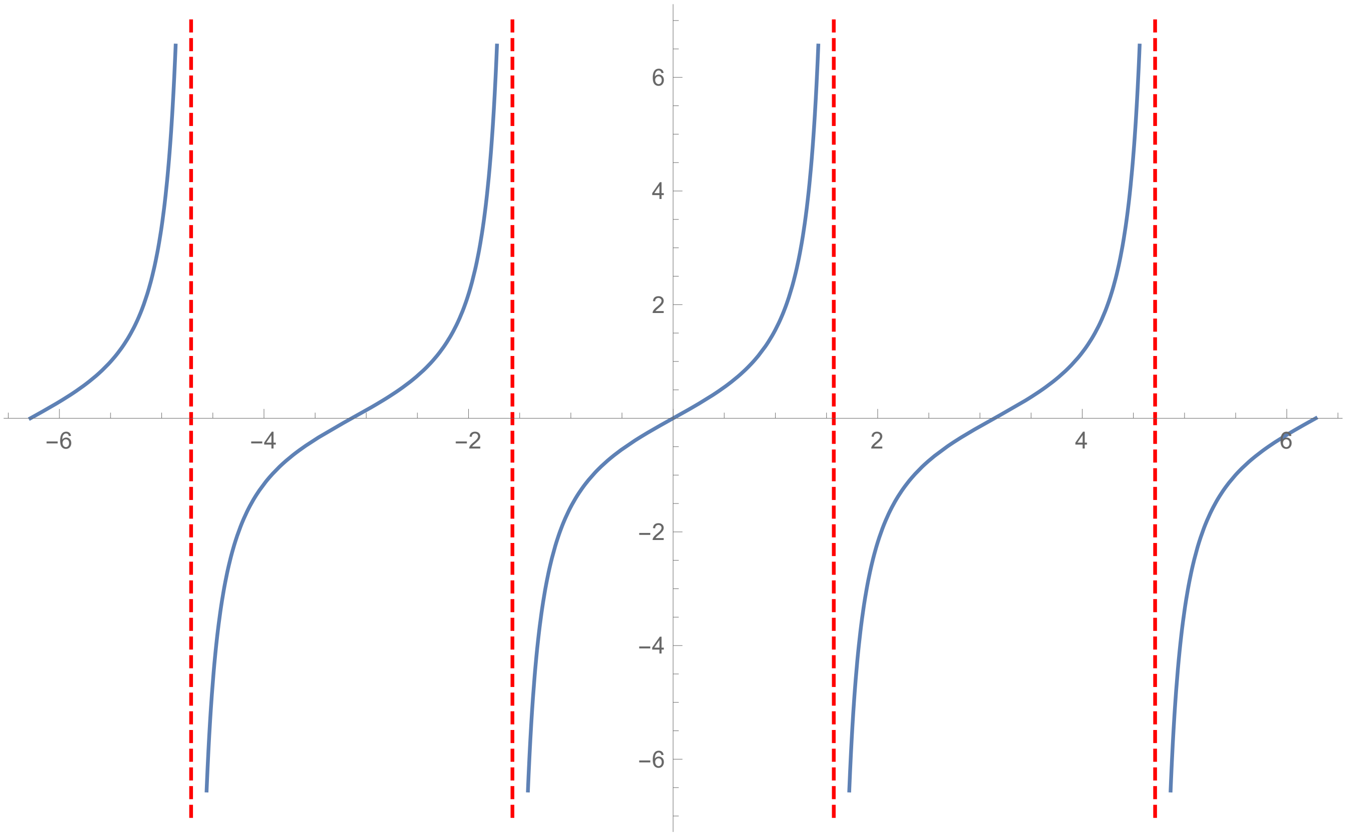
The graph of cosine is then



1.3. **Tangent.** Tangent is defined as $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. Looking at the values on the unit circle, $\tan\theta = \dfrac{x}{y}$. Here are some values of tangent:

$\theta =$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan\theta =$	undefined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Tangent takes positive values for angles in the first and third quadrant and negative values for angles in the second and fourth quadrant. The function $f(\theta) = \tan\theta$ has a period of π. It has vertical asymptotes at the values where $\cos\theta = 0$, which are the values $\theta = \dfrac{\pi}{2} + n\pi$ where n is an integer (these can also be described as odd multiples of $\dfrac{\pi}{2}$). The graph of tangent is



If $\theta = a$ is a vertical asymptote of $\tan\theta$, then we always have

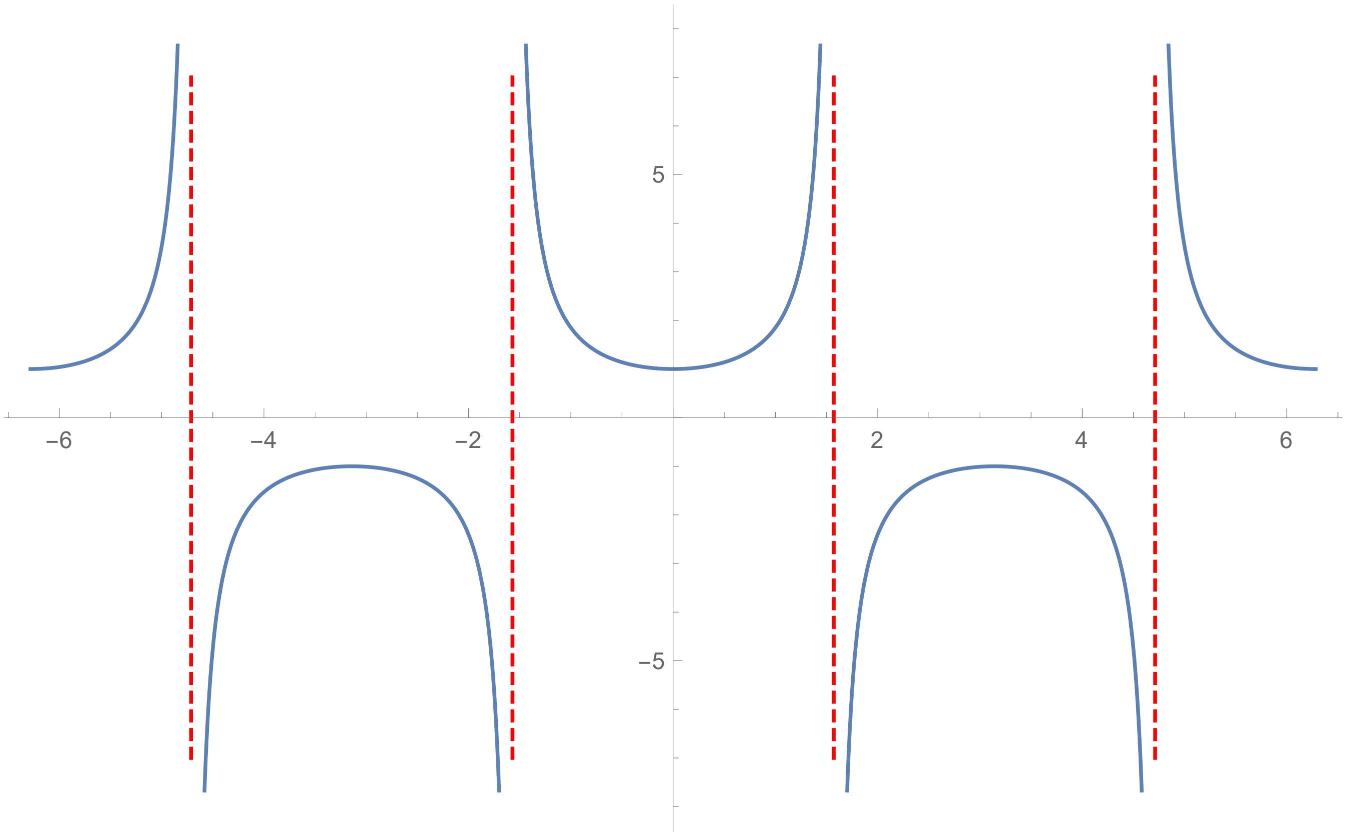
$$\lim_{\theta\to a^-} \tan\theta = \infty \quad \text{and} \quad \lim_{\theta\to a^+} \tan\theta = -\infty.$$

1.4. **Secant.** Secant is defined as the reciprocal of cosine, i.e., $\sec\theta = \dfrac{1}{\cos\theta}$. Here are some values of secant:

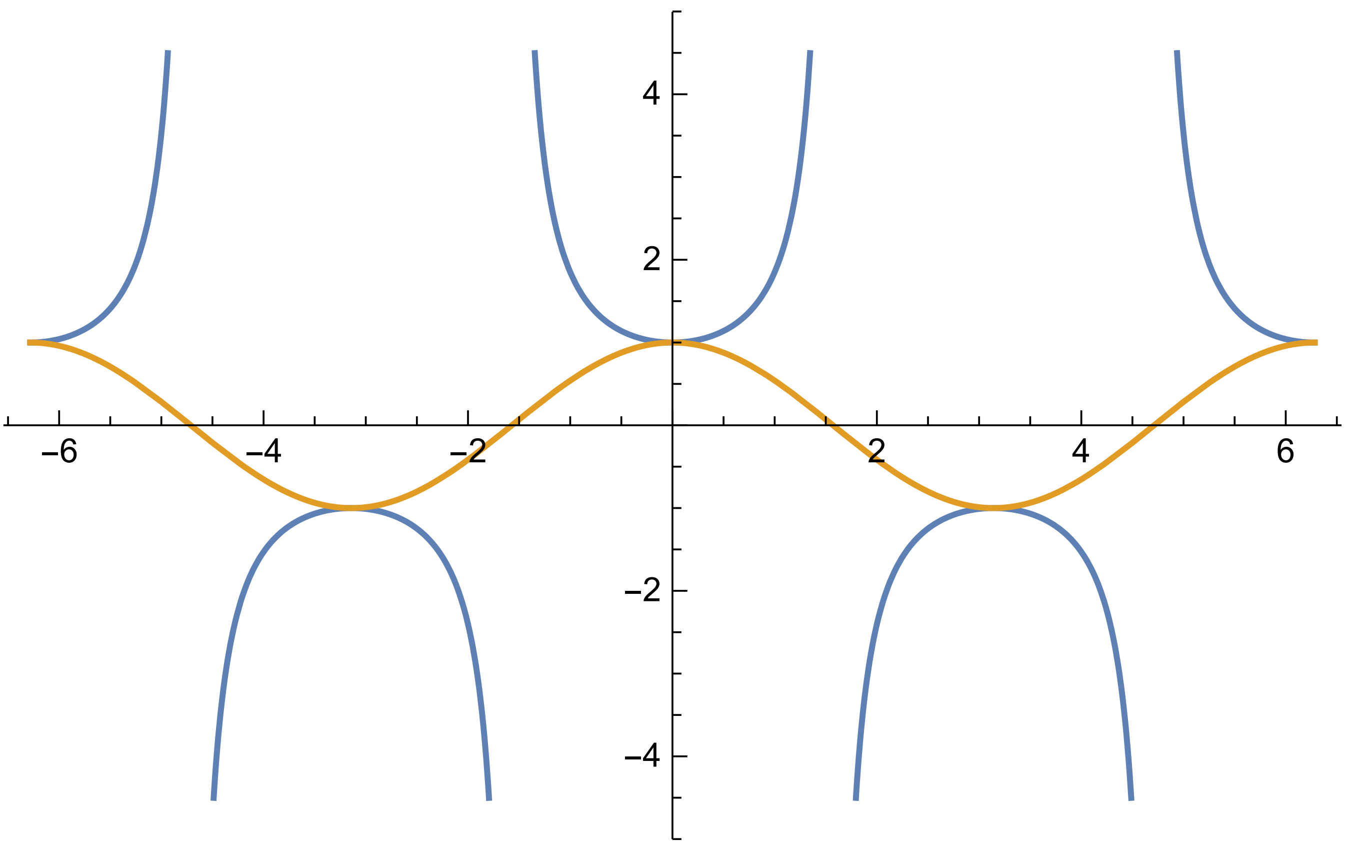
$\theta =$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sec\theta =$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined

$\theta =$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$\sec\theta =$	undefined	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	undefined

Secant is positive where cosine is, negative where cosine is, and is undefined when $\cos\theta = 0$. The function $f(\theta) = \sec\theta$ has a period of 2π. It has vertical asymptotes at the same places as tangent, that is where $\cos\theta = 0$. The graph of secant is



The graph of cosine can actually help you with graphing secant ($\sec\theta$ in blue, $\cos\theta$ in orange).



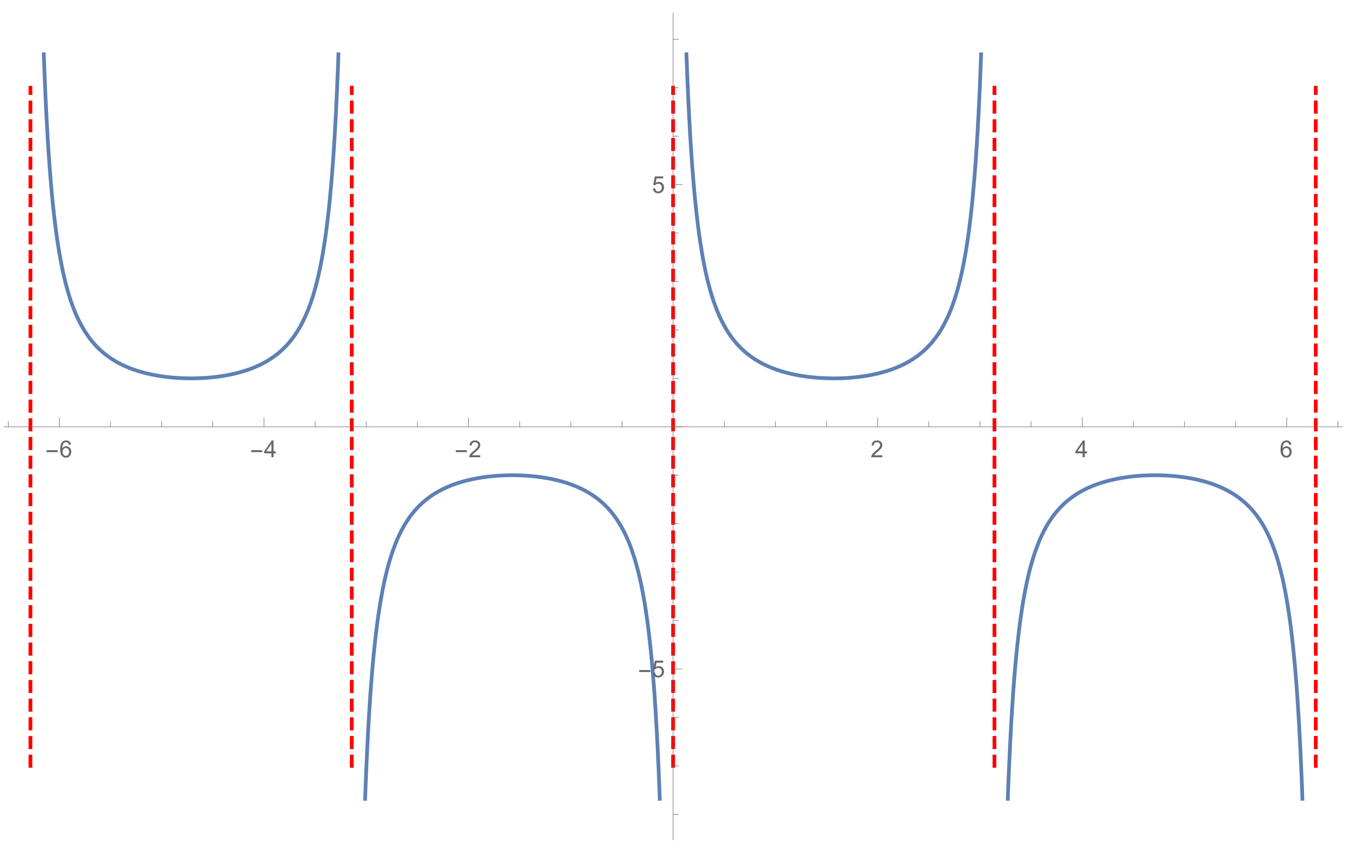
1.5. Cosecant.

Cosecant is defined as the reciprocal of sine, i.e., $\csc\theta = \dfrac{1}{\sin\theta}$. Here are some values of secant:

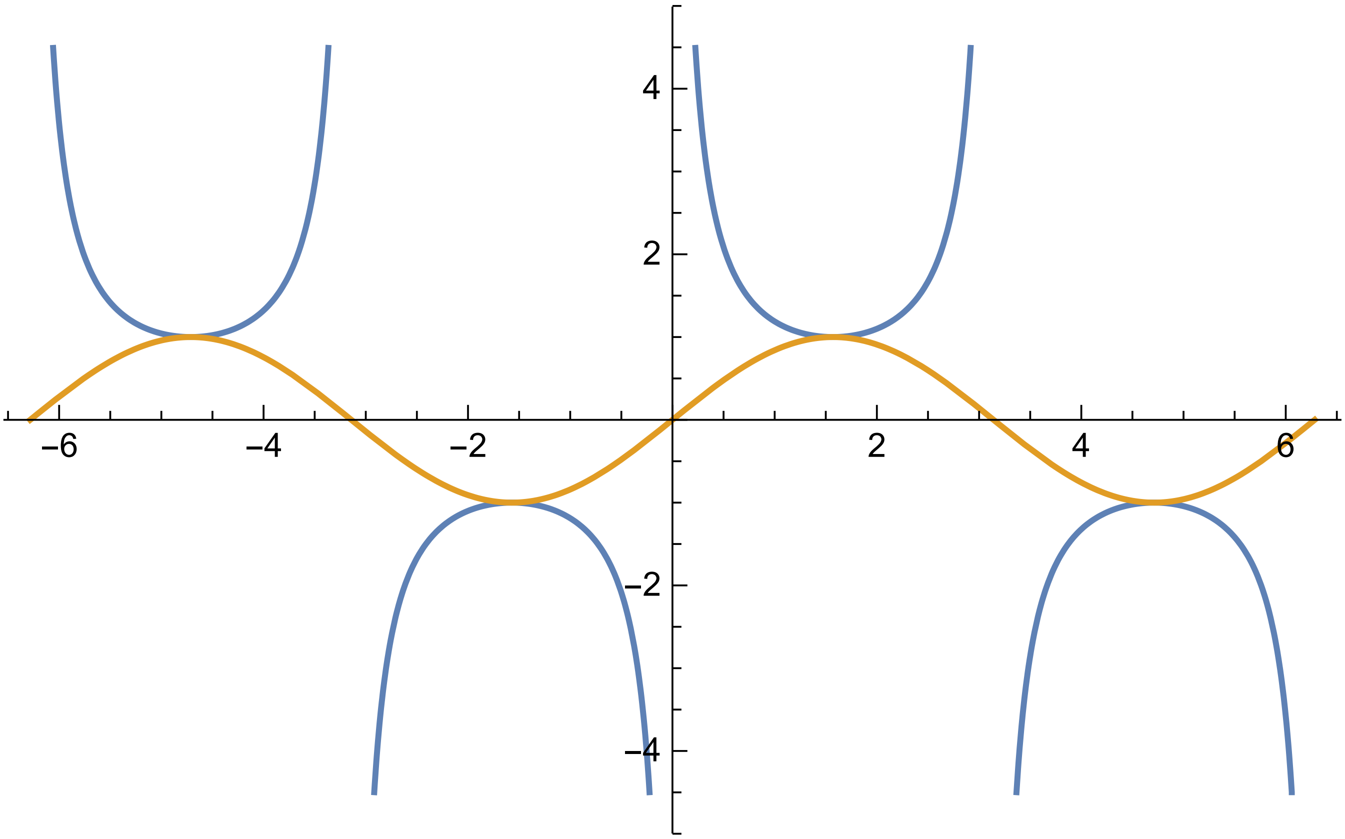
$\theta =$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\csc\theta =$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined

$\theta =$	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{3\pi}{4}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0
$\csc\theta =$	undefined	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	undefined

Cosecant is positive where sine is, negative where sine is, and is undefined when $\sin\theta = 0$. The function $f(\theta) = \csc\theta$ has a period of 2π. It has vertical asymptotes at the values where $\sin\theta = 0$, which are the values $\theta = n\pi$ where n is an integer. The graph of cosecant is



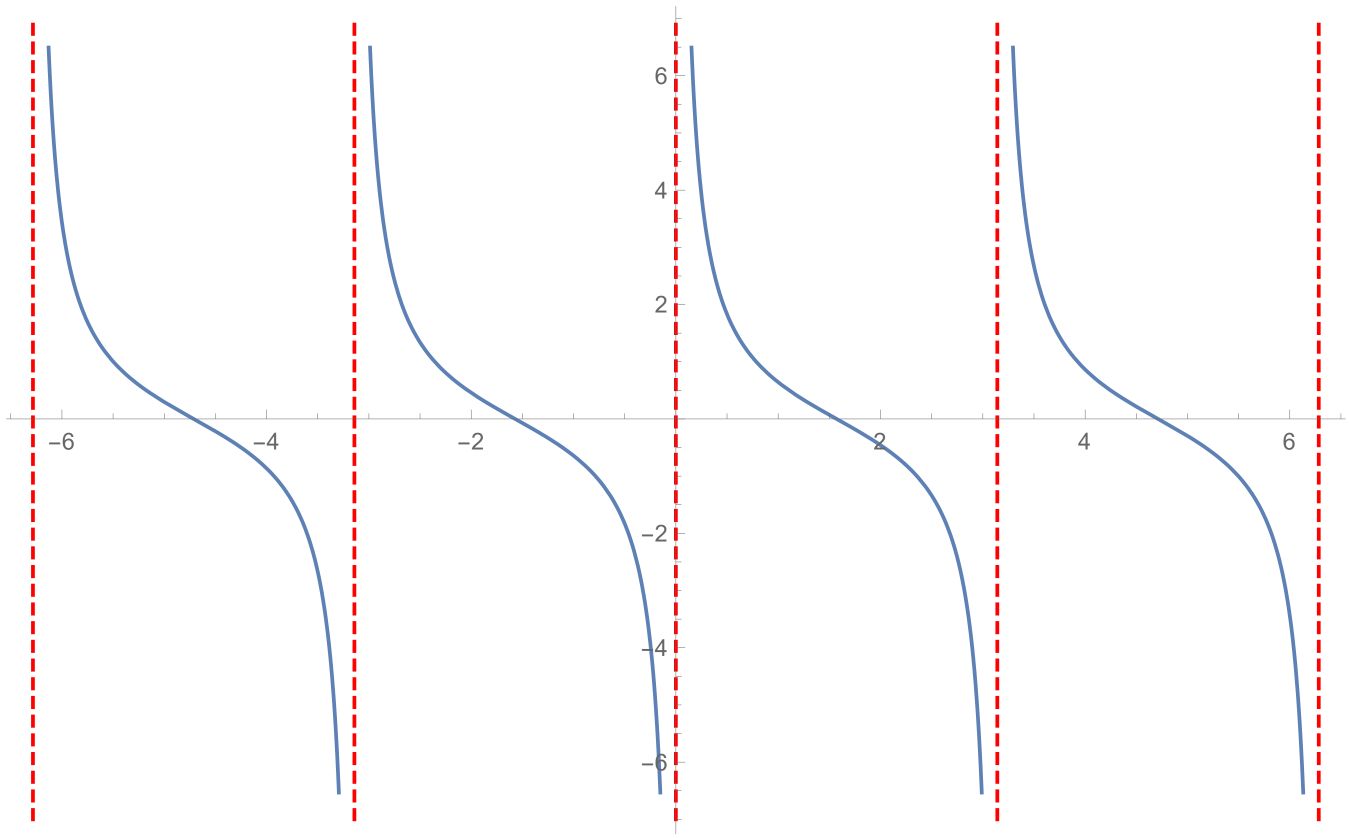
The graph of sine can actually help you with graphing secant ($\csc\theta$ in blue, $\sin\theta$ in orange).



1.6. **Cotangent.** The final of the six trig functions, cotangent, is defined as the reciprocal of tangent, or equivalently as the quotient of cosine and sine, i.e., $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$. Here are some values of cotangent:

$\theta =$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cot\theta =$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	undefined

Cotangent takes positive values for angles in the first and third quadrant and negative values for angles in the second and fourth quadrant, just like tangent. The function $f(\theta) = \tan\theta$ has a period of π. It has vertical asymptotes at the same places as $\sec\theta$, that is the values where $\sin\theta = 0$. The graph of cotangent is



If $\theta = a$ is a vertical asymptote of $\cot\theta$, then we always have

$$\lim_{\theta\to a^-} \cot\theta = -\infty \quad \text{and} \quad \lim_{\theta\to a^+} \cot\theta = \infty.$$

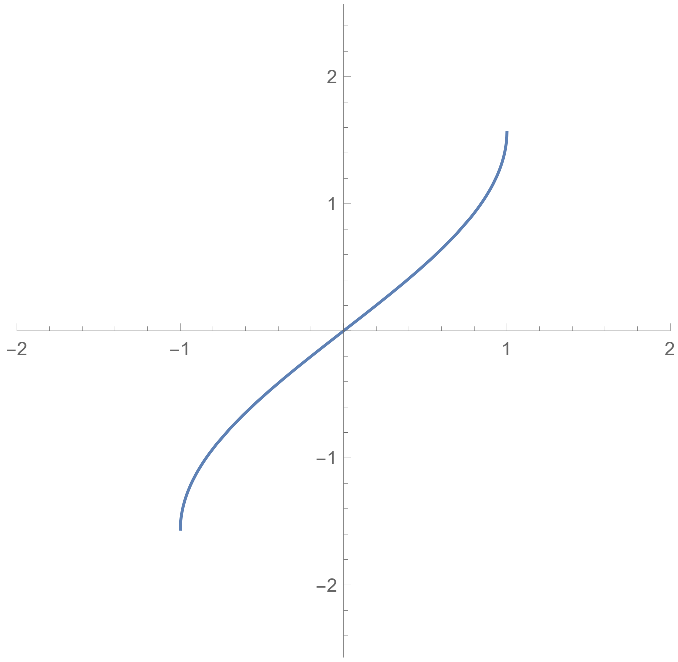
2. Inverse Trig Functions

2.1. Inverse Sine Function.

To come up with the inverse sine function, we need to restrict the domain of $\sin\theta$ so that it is one-to-one (i.e., passes the horizontal line test). We can accomplish this by restricting the domain of $\sin\theta$ to just $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Note that the range of $f(\theta) = \sin\theta$ on this interval is $[-1, 1]$. The inverse is written as $f^{-1}(x) = \arcsin x$ or $f^{-1}(x) = \sin^{-1} x$. Recall that

$$y = \arcsin x \quad \Longleftrightarrow \quad x = \sin y$$

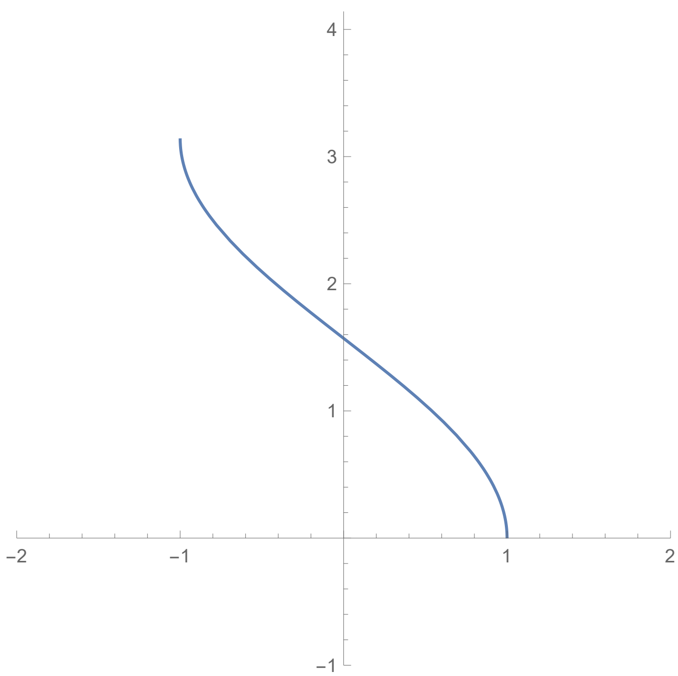
so the domain of $\arcsin x$ is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Below is a graph of the entire function



2.2. **Inverse Cosine Function.** To come up with the inverse cosine function, we need to restrict the domain of $\cos\theta$ so that it is one-to-one. This is accomplished by restricting the domain of $\cos\theta$ to $[0, \pi]$. Note that the range of $f(x) = \cos x$ on this interval is $[-1, 1]$. The inverse is written as $f^{-1}(x) = \arccos x$ or $f^{-1}(x) = \cos^{-1} x$. Recall that

$$y = \arccos x \qquad \Longleftrightarrow \qquad x = \cos y$$

so the domain of $\arccos x$ is $[-1, 1]$ and the range is $[0, \pi]$. Below is a graph of the entire function



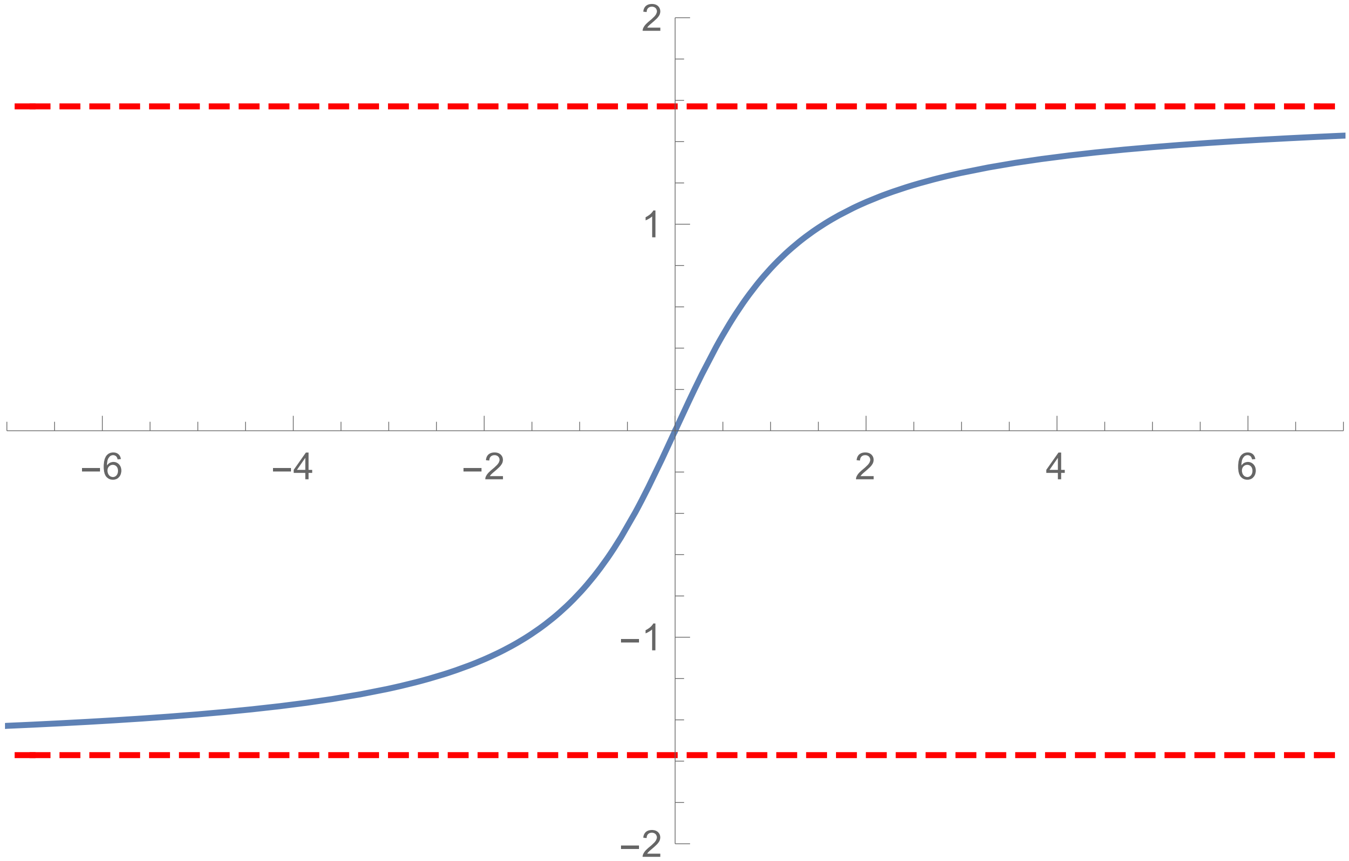
2.3. Inverse Tangent Function. To come up with the inverse tangent function, we need to restrict the domain of $\tan\theta$ so that it is one-to-one. This is accomplished by restricting the domain of $\tan\theta$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Note that the range of $f(\theta) = \tan\theta$ on this interval is $(-\infty, \infty)$. The inverse is written as $f^{-1}(x) = \arctan x$ or $f^{-1}(x) = \tan^{-1} x$. Recall that

$$y = \arctan x \qquad \Longleftrightarrow \qquad x = \tan y$$

so the domain of $\arctan x$ is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. One of the most interesting things about $\arctan x$ is that is has two different horizontal asymptotes.

$$\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2} \qquad \text{and} \qquad \lim_{x\to\infty} \arctan x = \frac{\pi}{2}.$$

Below is a graph of the entire function



3. Frequently Used Trig Identities

3.1. Pythagorean Identities.

(1) $\sin^2\theta + \cos^2\theta = 1$

(2) $1 + \tan^2\theta = \sec^2\theta$

(3) $1 + \cot^2\theta = \csc^2\theta$

3.2. Double-Angle Formulas.

(1) $\sin 2u = 2\sin u \cos u$

(2) $\cos 2u = \cos^2 u - \sin^2 u$

3.3. Power-Reducing Formulas.

(1) $\sin^2 u = \dfrac{1 - \cos 2u}{2}$

(2) $\cos^2 u = \dfrac{1 + \cos 2u}{2}$

3.4. Sum and Difference Formulas.

(1) $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$

(2) $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$

3.5. Even/Odd Identities.

(1) $\sin(-x) = -\sin x$

(2) $\cos(x) = \cos x$

(3) $\tan(-x) = -\tan x$

(4) $\sec(-x) = \sec x$

(5) $\csc(-x) = -\csc x$

(6) $\cot(-x) = -\cot x$