1 Regions in $\mathbb{R}^2$

1.1 Mass

Recall that, for constant density regions in the plane, the mass of the region is given by

$$m = \rho A$$

where m is the mass, ρ is the density, and A is the area of the region. Of course not all regions need have constant density. So to find the mass of a region we integrate the density function over the region, that is, for a region R in the plane with density function $\rho(x, y)$, the mass of R is

$$m = \iint_R \rho \, dA$$

1.2 Moments and Center of Mass

We will consider the moments of mass of a region (or lamina) in the plane about the x- and y-axes of variable density.

Recall that the moment of mass of a point mass in $\mathbb{R}^2$ about the x-axis is the product of the mass of the point and the distance from the point to the x-axis. For a system of particles, the moment of mass of the system about the x-axis is the sum of the moments of each of the individual points. When defining the moments of mass of these regions, we will take a similar approach, which is the following:

First we break up the region into a bunch of tiny squares. Then we assume that the mass in each rectangle R_i is concentrated at some point (x_i, y_i) inside that rectangle. Suppose that the mass of the rectangle R_i is ΔA_i. Then we can treat this like we are finding the moment of mass of a system of particles. So, the moment of mass of R_i about the x-axis is:

$$(mass)(y_i) \approx [\rho(x_i, y_i) \Delta A_i](y_i),$$

and similarly the moment of mass of R_i about the y-axis is

$$(mass)(x_i) \approx [\rho(x_i, y_i) \Delta A_i](x_i).$$

Now, to get the true moments of mass what we do is let the rectangles keep getting smaller and smaller, thus giving us the following

Definition 1 (Moment and Center of Mass) *Let ρ be a continuous density function on the planar lamina R. The* moments of mass *with respect to the x- and y- axes are*

$$M_x = \iint_R y\rho(x, y) \; dA$$

and

$$M_y = \iint_R x\rho(x, y) \; dA$$

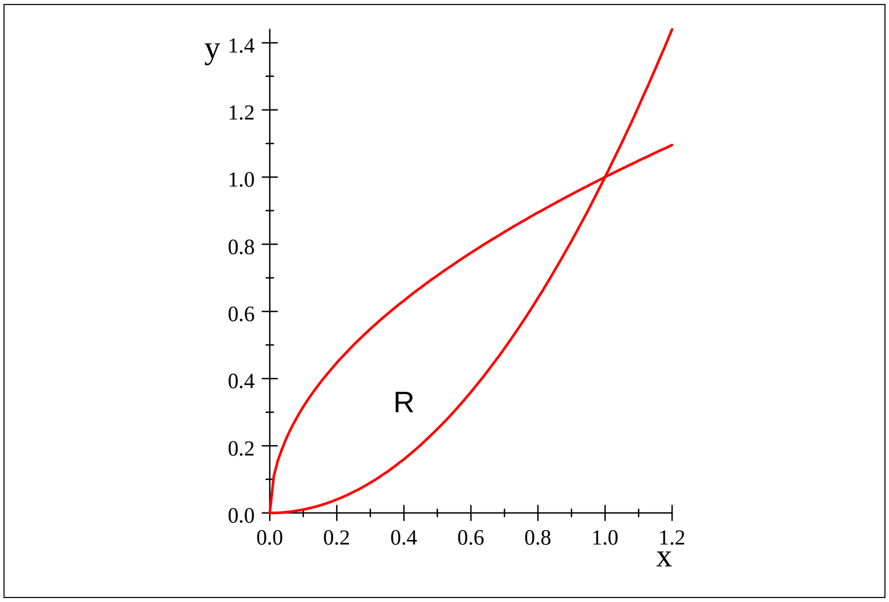
respectively. If m is the mass of R, then the center of mass is given by $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}.$$

Let's compute a couple examples.

Example 2 *Find the center of mass of the region bounded by the curves $y = x^2$ and $x = y^2$ with density function $\rho(x, y) = y + 1$.*

Solution 3 *The region in question here is:*



First we need to find the moments of mass about the two axes, and before that, we need to find the mass of the region. First, the mass

$$\begin{aligned} m &= \iint_R \rho\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (y+1)\, dydx = \int_0^1 \left[\left(\frac{1}{2}x + x^{\frac{1}{2}}\right) - \left(x^4 + x^2\right)\right] dx \\ &= \int_0^1 \left(x^{\frac{1}{2}} + \frac{1}{2}x - x^2 - \frac{1}{2}x^4\right) dx = \frac{2}{3} + \frac{1}{4} - \frac{1}{3} - \frac{1}{10} = \frac{29}{60} \end{aligned}$$

Now we compute the two moments:

- *about the x-axis*

$$\begin{aligned} M_x &= \iint_R y\rho\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} y\,(y+1)\, dydx = \int_0^1 \int_{x^2}^{\sqrt{x}} \left(y^2 + y\right) dydx \\ &= \int_0^1 \left(\frac{1}{3}x^{\frac{3}{2}} + \frac{1}{2}x - \frac{1}{3}x^6 - \frac{1}{2}x^4\right) dx = \frac{2}{15} + \frac{1}{4} - \frac{1}{21} - \frac{1}{10} = \frac{33}{140} \end{aligned}$$

- *about the y-axis*

$$\begin{aligned} M_y &= \iint_R x\rho\, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} x\,(y+1)\, dydx = \int_0^1 \int_{x^2}^{\sqrt{x}} (xy + x)\, dydx \\ &= \int_0^1 \left(\frac{1}{2}x^2 + x^{\frac{3}{2}} - x^3 - \frac{1}{2}x^5\right) dx = \frac{1}{6} + \frac{2}{5} - \frac{1}{4} - \frac{1}{12} = \frac{7}{30} \end{aligned}$$

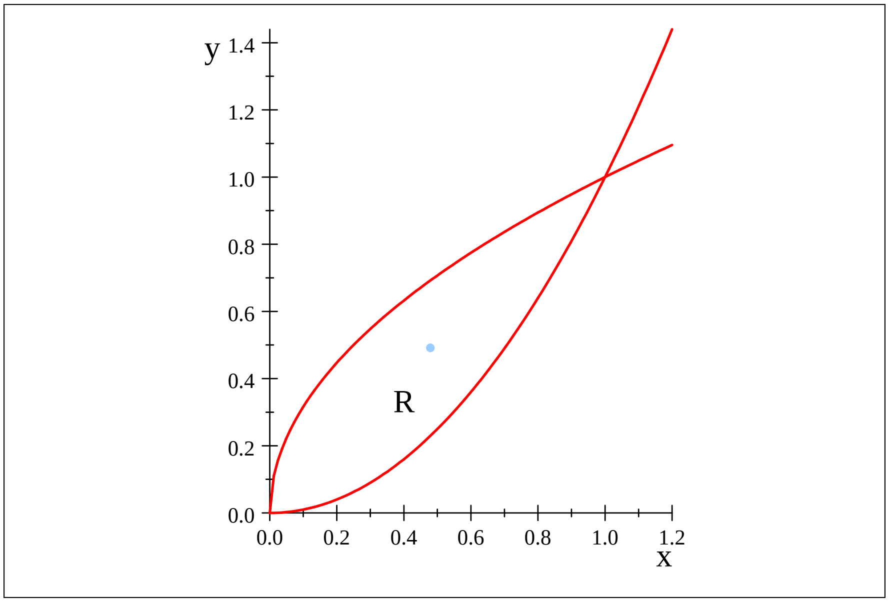
Finally we can compute the center of mass $(\bar{x}, \bar{y})$:

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{7}{30}}{\frac{29}{60}} = \frac{7}{30}\frac{60}{29} = \frac{14}{29}$$

and

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{33}{140}}{\frac{29}{60}} = \frac{33}{140}\frac{60}{29} = \frac{99}{203}$$

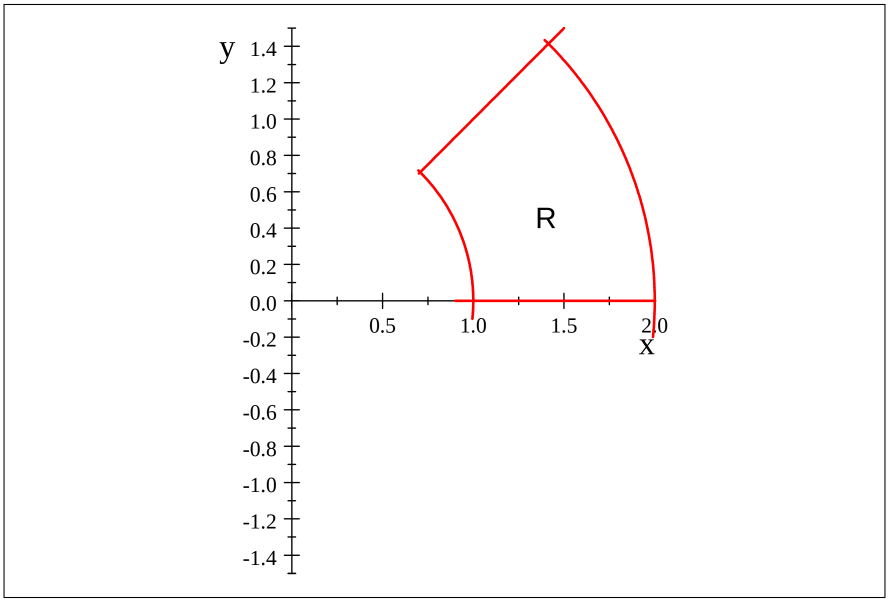
Here is the center of mass graphically:



Now, let's see another example.

Example 4 *Find the center of mass of the region bounded by* $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $y = x$, *and* $y = 0$, *where* $x \geq 0$ *with density function* $\rho(x, y) = \arctan\left(\frac{y}{x}\right)$.

Solution 5 *The region in question is*



First let's find the mass

$$m = \iint_R \rho \, dA = \iint_R \arctan\left(\frac{y}{x}\right) \, dA = \int_0^{\frac{\pi}{4}} \int_1^2 \theta r dr d\theta = \frac{3}{2} \int_0^{\frac{\pi}{4}} \theta \, d\theta = \frac{3\pi^2}{64}$$

and now the moments:

$$\begin{aligned} M_x &= \iint_R y\rho \, dA = \int_0^{\frac{\pi}{4}} \int_1^2 (r \sin\theta)\theta r \, dr d\theta = \int_0^{\frac{\pi}{4}} \int_1^2 r^2 \theta \sin\theta \, dr d\theta \\ &= \frac{7}{3} \int_0^{\frac{\pi}{4}} \theta \sin\theta \, d\theta \overset{IBP}{=} \frac{7}{3} \left(-\theta \cos\theta + \sin\theta\right)\Big|_0^{\frac{\pi}{4}} = \frac{7}{3} \left(-\frac{\pi}{4} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) \\ &= \frac{7\sqrt{2}\,(4 - \pi)}{24} \end{aligned}$$

and

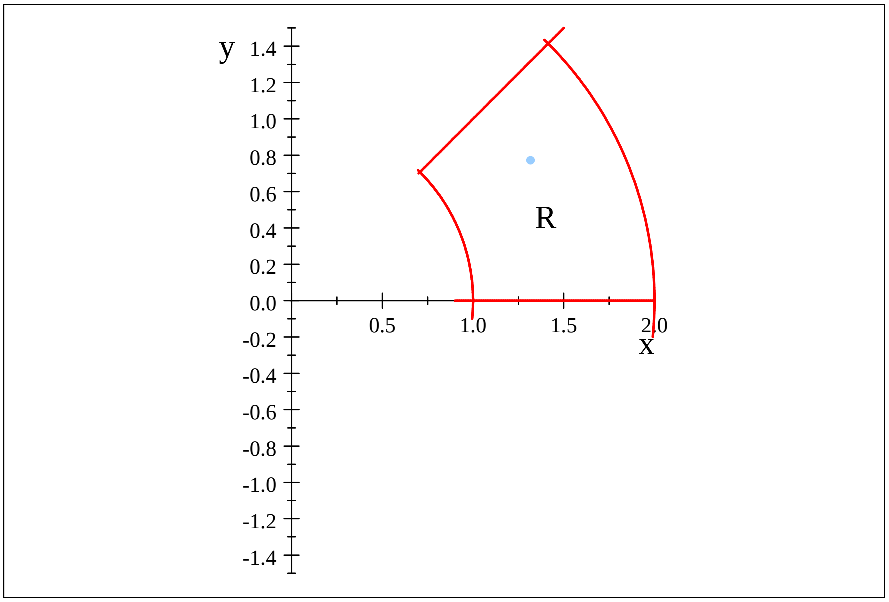
$$\begin{aligned} M_y &= \iint_R x\rho \, dA = \int_0^{\frac{\pi}{4}} \int_1^2 (r \cos\theta)\theta r \, dr d\theta = \int_0^{\frac{\pi}{4}} \int_1^2 r^2 \theta \cos\theta \, dr d\theta \\ &= \frac{7}{3} \int_0^{\frac{\pi}{4}} \theta \cos\theta \, d\theta \overset{IBP}{=} \frac{7}{3} \left(\theta \sin\theta + \cos\theta\right)\Big|_0^{\frac{\pi}{4}} = \frac{7}{3} \left(\frac{\pi}{4} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 \right) \\ &= \frac{7}{3} \left[\frac{\sqrt{2}\,(4 + \pi) - 8}{8} \right] \end{aligned}$$

So, the center of mass, $(\bar{x}, \bar{y})$, *is given by:*

$$\bar{x} = \frac{M_y}{m} = \frac{7}{3}\left[\frac{\sqrt{2}\,(4+\pi)-8}{8}\right]\frac{64}{3\pi^2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{7\sqrt{2}\,(4-\pi)}{24}\frac{64}{3\pi^2} = \frac{56\sqrt{2}\,(4-\pi)}{9\pi^2}$$

Graphically, here is the center of mass.



1.3 Moments of Inertia

The moment of inertia of a region measures its resistance to rotating about a given axis. For a particle of mass m, its moment of inertia about a given axis is mr^2, where r is the distance from the particle to the axis. Adopting the same methods as above, we can define:

- The *moment of inertia* of a lamina R *about the x-axis*

$$I_x = \iint_R y^2 \rho(x,y)\ dA$$

- The *moment of inertia* of a lamina R *about the y-axis*

$$I_y = \iint_R x^2 \rho(x,y)\ dA$$

2 Regions in $\mathbb{R}^3$

Everything is basically the same as in the 2 variable case, except now we have a third varible floating around. So now our density function is a function $\rho(x,y,z)$, and this time we will be finding moments of mass about the *coordinate planes* in $\mathbb{R}^3$. To derive these equations, you go through the same thing we did for planar regions, but instead of using rectangles, you use boxes. Rather than deriving the formulas again, we will just state them. Let D be our region, and let ρ be our density function.

- Mass

$$m = \iiint_D \rho \; dV$$

- moment about the yz-plane

$$M_{yz} = \iiint_D x\rho \; dV$$

- moment about the xz-plane

$$M_{xz} = \iiint_D y\rho \; dV$$

- moment about the xy-plane

$$M_{xy} = \iiint_D z\rho \; dV$$

- center of mass $(\bar{x}, \bar{y}, \bar{z})$

$$\begin{aligned} \bar{x} &= \frac{M_{yz}}{m} \\ \bar{y} &= \frac{M_{xz}}{m} \\ \bar{z} &= \frac{M_{xy}}{m} \end{aligned}$$

Example 6 *Find the center of mass of the region bounded by $x^2 + y^2 = 9$, $z = 2$, and $z = 4$, whose density at any point is proportional to the square of the distance to that point from the origin.*

Solution 7 *First, notice that the density function is $\rho(x, y, z) = k(x^2 + y^2 + z^2)$ for some constant k. First, let's compute the mass:*

$$\begin{aligned} m &= \iiint_D k\left(x^2 + y^2 + z^2\right) dV = k \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{2}^{4} \left(x^2 + y^2 + z^2\right) \; dzdydx \\ &= k \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(2x^2 + 2y^2 + \frac{56}{3}\right) \; dydx \end{aligned}$$

Before proceeding further with this integral, notice that we are back to a double integral over a circle, so at this point, we might as well switch to polar coordinates, doing this, we get

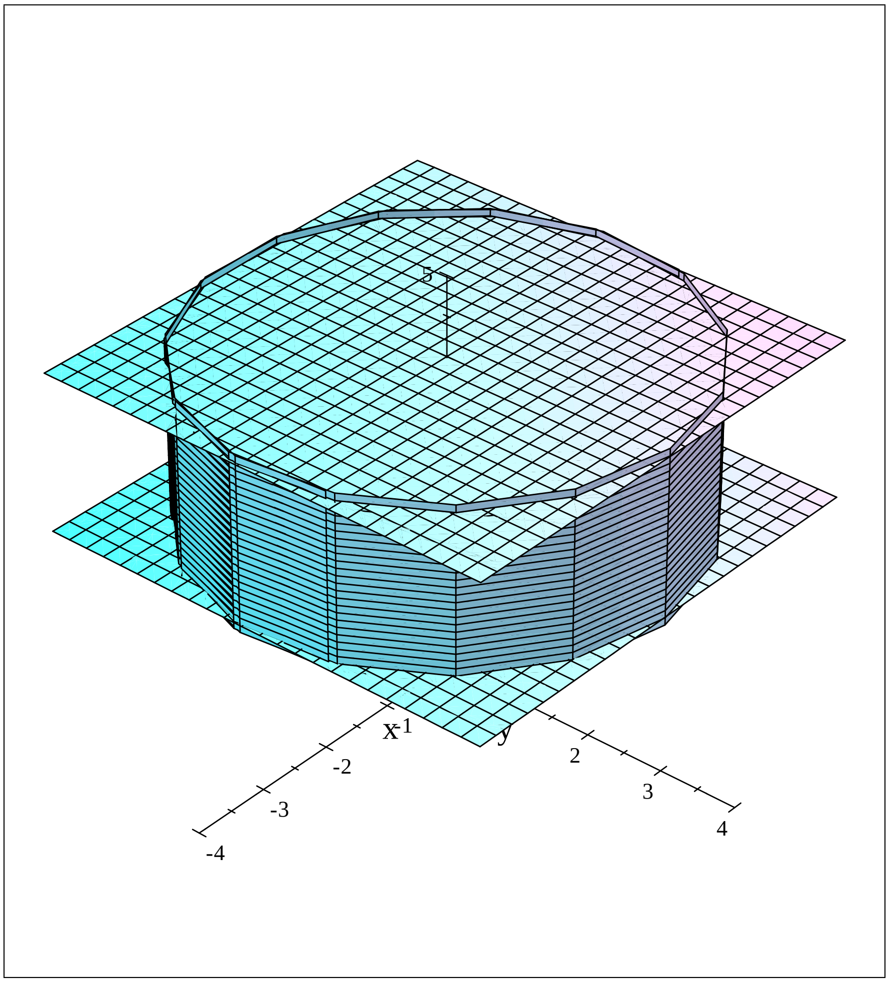
$$\begin{aligned} m &= k \int_{0}^{2\pi} \int_{0}^{3} \left(2r^2 + \frac{56}{3}\right) rdrd\theta = k \int_{0}^{2\pi} \int_{0}^{3} \left(2r^3 + \frac{56}{3}r\right) drd\theta \\ &= k \int_{0}^{2\pi} \frac{249}{2} d\theta = k\frac{249}{2}(2\pi) = 249k\pi \end{aligned}$$

We did the switch to polar coordinates after we had already reduced it to a double integral, but notice that we could have done it beforehand as well, that is, we could have done the integral

$$k \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{2}^{4} \left(x^2 + y^2 + z^2\right) \; dzdydx = k \int_{0}^{2\pi} \int_{0}^{3} \int_{2}^{4} \left(r^2 + z^2\right) \; rdzdrd\theta$$

instead. Doing this is switching to cylindrical coordinates *which you will learn about tomorrow. Basically, cylindrical coordinates are just polar coordinates with a z added on.*

Now, let's examine the region and the density function a bit.



Notice that the region is rotationally symmetric about the z-axis. Also notice that the density function is rotationally symmetric about the z-axis as well (in fact, since it's level surfaces are spheres, it is rotationally symmetric about any line through the origin). What this means for us is that we expect for the center of mass to lie on the z-axis, in other words, we expect $\bar{x} = \bar{y} = 0$ (and hence $M_{yz} = M_{xz} = 0$ as well). So we really only need to compute M_{xy} (a bit of a relief). So, let's compute it

$$\begin{aligned} M_{xy} &= k \iiint_D z\left(x^2+y^2+z^2\right) dV = k \int_0^{2\pi} \int_0^3 \int_2^4 z\left(r^2+z^2\right) r dz dr d\theta \\ &= k \int_0^{2\pi} \int_0^3 \int_2^4 \left(r^3 z + r z^3\right) dz dr d\theta = k \int_0^{2\pi} \int_0^3 \left(6r^3 + 60r\right) dr d\theta \\ &= k \int_0^{2\pi} \frac{783}{2} d\theta = 783 k\pi \end{aligned}$$

Thus

$$\bar{z} = \frac{M_{xy}}{m} = \frac{783k\pi}{249k\pi} = \frac{261}{83}$$

so that the center of mass is

$$\left(0, 0, \frac{261}{83}\right).$$

2.1 Moments of Inertia

As in the 2 dimensional case, we can also define the moments of inertia for a region D about each of the coordinate axes. Note, for example, that the smallest distance from any point (x, y, z) to, say the z-axis, is given by $\sqrt{x^2 + y^2}$. So the *moments of inertia* of D about

- *the x-axis*

$$I_x = \iiint_D \left(y^2 + z^2\right) \rho\left(x, y, z\right)\ dV$$

- *the y-axis*

$$I_y = \iiint_D \left(x^2 + z^2\right) \rho\left(x, y, z\right)\ dV$$

- *the z-axis*

$$I_z = \iiint_D \left(x^2 + y^2\right) \rho\left(x, y, z\right)\ dV$$