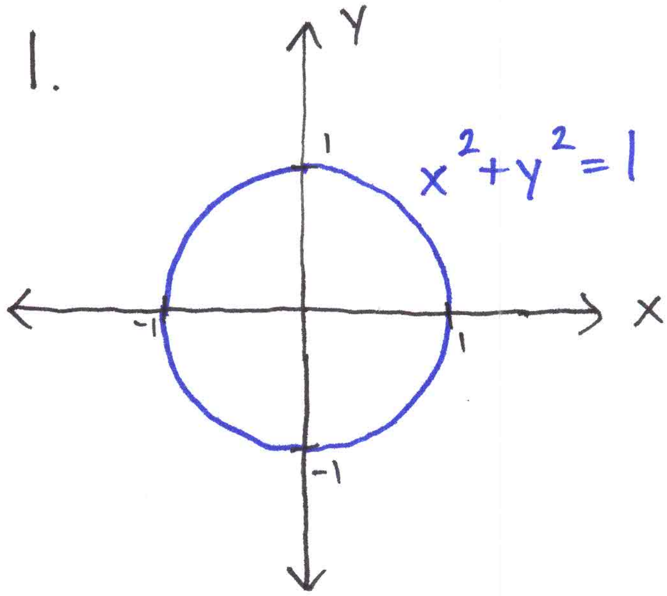
Calc III - February 21, 2013

Level Curves: A level curve of a function of $\boxed{\text{two}}$ variables, $z=f(x,y)$, is the set of (x,y) such that $f(x,y)=c$ for some fixed number c. That is, we pick a specific value of z, and find all (x,y) such that $f(x,y)$ is this value. This value (c) should be in the range of f.

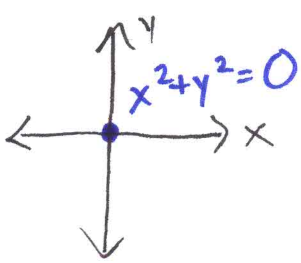
Ex: $f(x,y)=x^2+y^2$

Let's look at 3 level curves of f:

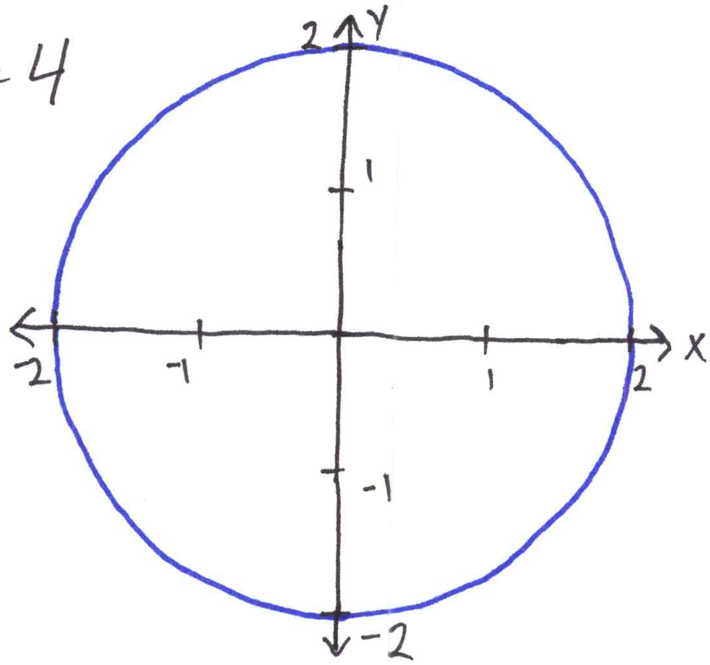
a) $f(x,y)=1$.



b) $f(x,y)=0$



c) $f(x,y)=4$

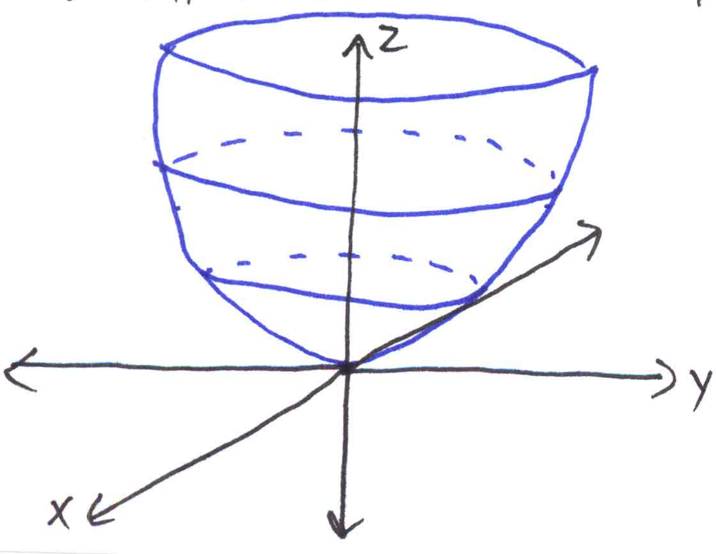


Level Surfaces: A level surface of a function of $\boxed{\text{three}}$ variables, $f(x,y,z)$, is the set of (x,y,z) such that $f(x,y,z)=c$ for some fixed number c in the range of f.

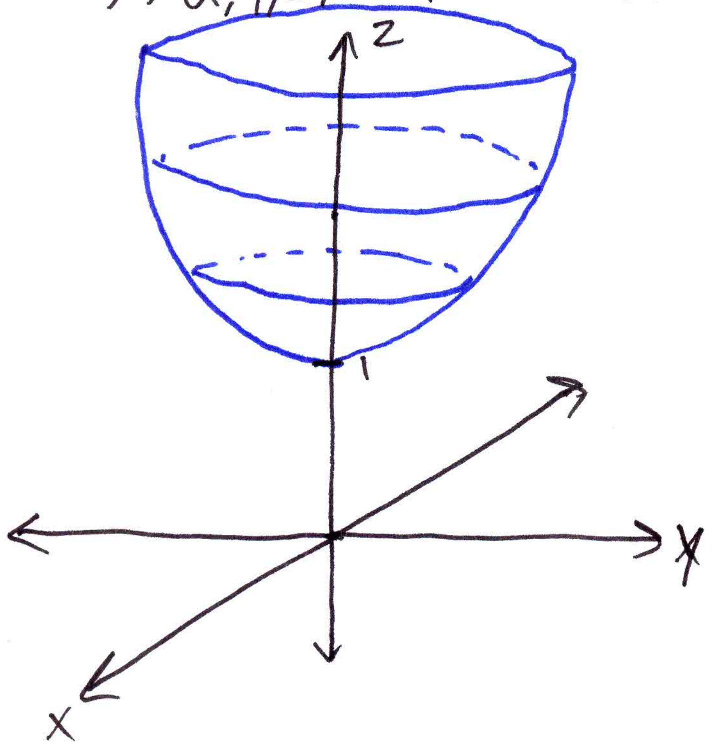
Ex: $f(x,y,z)=x^2+y^2-z$

Let's look at 3 level surfaces:

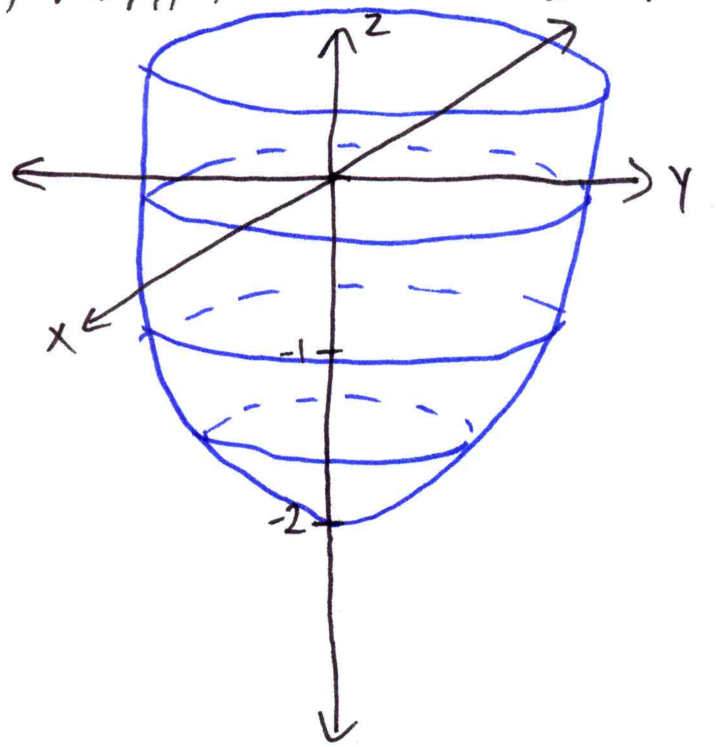
a) $f(x,y,z)=0 \iff 0=x^2+y^2-z \iff z=x^2+y^2$



b) $f(x,y,z) = -1 \iff z = x^2 + y^2 + 1$



c) $f(x,y,z) = 2 \iff z = x^2 + y^2 - 2$



Curve vs. Surface

A curve is "one dimensional" in the sense that if you stand at any point on a curve, the are only two directions you can move. On the other hand, on a surface, there are a "circles worth" (think of a compass) of directions you can move. So a surface is "two dimensional" in this sense.

Level Curves vs. Level Surfaces

Aside from the definition of curve and surface above, what else distinguishes a level curve from a level surface? The key is the number of variables in the domain of the function.

$\boxed{2}$ variables $\rightsquigarrow$ level $\boxed{\text{curve}}$

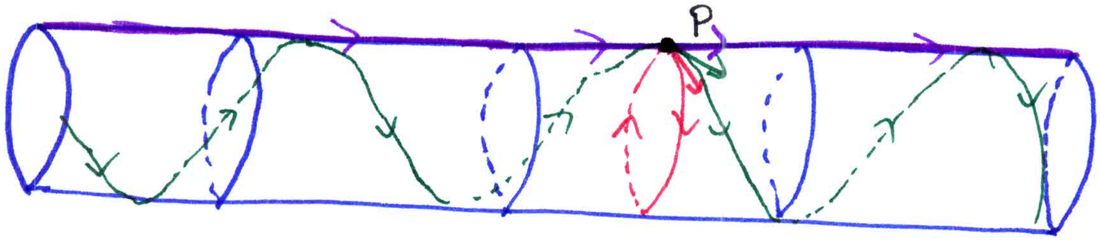
$\boxed{3}$ variables $\rightsquigarrow$ level $\boxed{\text{surface}}$

Just as a plane is "the next dimension* up" from a line, we can think as a surface as "the next dimension* up" from a curve.

Tangent Planes

Curves have tangent lines, so, by the analogy, a surface should have a tangent plane.

How do we get tangent vectors to a surface? Let's see an example:



The idea is, to get a tangent vector to the surface at p, we look at the tangent vector to a curve passing through p.

Formally: Let's look at a level surface given by $f(x,y,z)=c$. A curve in this surface has the form $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$. Plug this in: $f(x(t),y(t),z(t))=c$. Take the derivative with respect to t using the chain rule to get:

$$\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}=0 \qquad (*)$$

Remember that $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

Then, we can rewrite equation (*) as:

$$\nabla f \cdot \vec{r}\,'(t) = 0$$

This means any tangent vector is perpendicular to ∇f at that point. Let $\vec{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, then the tangent plane to S at the point $P = (x_0, y_0, z_0)$ is the plane passing through P with normal vector given by $\nabla f(x_0, y_0, z_0)$.

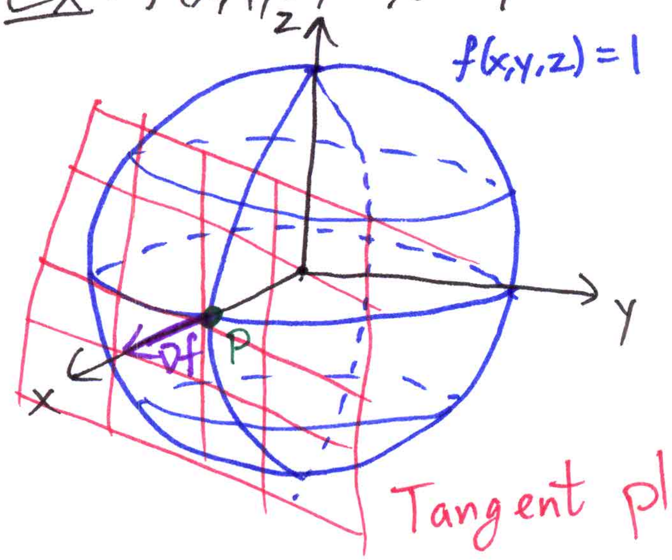
That is, the equation for the plane is:

$$\boxed{\nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0}$$

-or-

$$\boxed{f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0}$$

<u>Ex</u>: $f(x,y,z) = x^2 + y^2 + z^2$



$P = (1,0,0)$

∇f perpendicular to tangent plane

$$\nabla f = \langle 2x, 2y, 2z \rangle, \quad \nabla f(1,0,0) = \langle 2,0,0 \rangle$$

So, the equation for the tangent plane at $(1,0,0)$

is $\langle 2,0,0 \rangle \cdot \langle x-1, y-0, z-0 \rangle = 2(x-1) = 0$

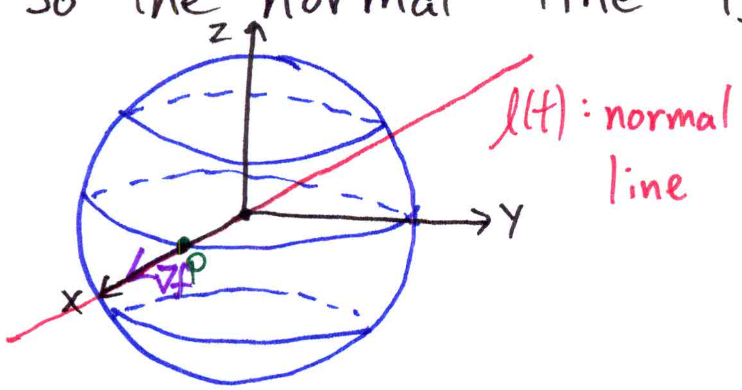
-or- $x = 1$.

<u>Normal Lines</u>: Notice that the gradient vector was perpendicular to the level surface at P. Since we have a vector $(\nabla f(x_0, y_0, z_0))$ and a point $(P = (x_0, y_0, z_0))$ we can make a line perpendicular to S at P, called the <u>normal line to S at P</u>. The equation for the line

is: $\boxed{\ell(t) = \langle x_0, y_0, z_0 \rangle + t\, \nabla f(x_0, y_0, z_0)}$

In the previous example, our point was $P = (1,0,0)$ and our normal vector was $\nabla f(1,0,0) = \langle 2,0,0 \rangle$

So the normal line is: $\ell(t) = \langle 1,0,0 \rangle + t \langle 2,0,0 \rangle$
$= \langle 1+2t, 0, 0 \rangle$



Example Find the tangent plane and normal line to the surface $z=-x^2+y^2$ at the point $(0,-1,1)$.

Solution: First we notice that $z=-x^2+y^2$ is a level surface of $f(x,y,z)=-x^2+y^2-z$, specifically $f(x,y,z)=0$. So $\nabla f=\langle -2x, 2y, -1\rangle$, and

$\nabla f(0,-1,1)=\langle 0,-2,-1\rangle$

Thus, the tangent plane is:

$$\langle 0,-2,-1\rangle \cdot \langle x-0, y+1, z-1\rangle$$

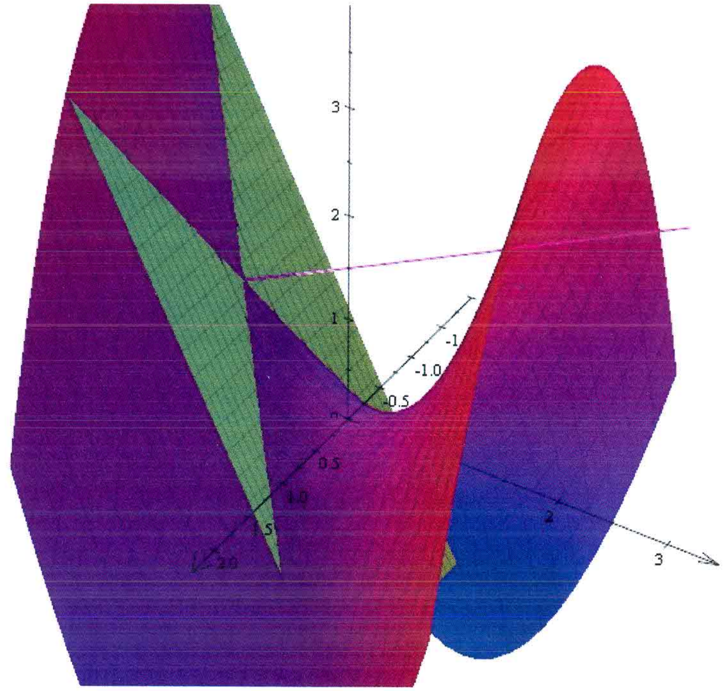
$$= -2(y+1)-(z-1)=0$$

-or- $\boxed{-2y-z=1}$ -or- $\boxed{2y+z=-1}$

and the normal line is:

$$\boxed{\ell(t)=\langle 0,-1,1\rangle + t\langle 0,-2,-1\rangle = \langle 0,-1-2t,1-t\rangle}$$

View from first octant



View from quadrant where x and z are non-negative and y is non-positive