

## 1/24 - Extra Problems

① Show that  $\vec{b} - \text{proj}_{\vec{a}} \vec{b}$  is orthogonal to  $\vec{a}$ .  
( $\vec{a} \neq \vec{0}$ ).

Solution

$$\vec{b} - \text{proj}_{\vec{a}} \vec{b} = \vec{b} - \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$$

$$(\vec{b} - \text{proj}_{\vec{a}} \vec{b}) \cdot \vec{a} = \vec{b} \cdot \vec{a} - \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} (\vec{a} \cdot \vec{a}) = \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{a} = 0$$

So  $\vec{b} - \text{proj}_{\vec{a}} \vec{b} \perp \vec{a}$ .

② Show  $|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|$ .

Solution

Recall that  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$  where  $\theta$  is the angle between  $\vec{a}$  &  $\vec{b}$ . Then

$$|\vec{a} \cdot \vec{b}| = ||\vec{a}| |\vec{b}| \cos \theta| = |\vec{a}| |\vec{b}| |\cos \theta|$$

Since  $|\cos \theta| \leq 1$  we have

$$|\vec{a} \cdot \vec{b}| = |\vec{a}| |\vec{b}| |\cos \theta| \leq |\vec{a}| |\vec{b}|$$

③ Show  $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

$$\begin{aligned} |\vec{a} + \vec{b}|^2 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} \\ &= |\vec{a}|^2 + 2|\vec{a}| |\vec{b}| \cos \theta + |\vec{b}|^2 \\ &\leq |\vec{a}|^2 + 2|\vec{a}| |\vec{b}| + |\vec{b}|^2 \quad (\cos \theta \leq 1) \\ &= (|\vec{a}| + |\vec{b}|)^2 \end{aligned}$$

Since  $|\vec{a} + \vec{b}|$  and  $|\vec{a}| + |\vec{b}|$  are non-negative, taking the square root of both sides does not change the

inequality, so

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$$

④ Show  $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

Solution

Recall  $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$

&  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$

Square these equations and add them together:

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta$$

$$+ (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 \cos^2 \theta$$

$$\underline{|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 (\sin^2 \theta + \cos^2 \theta) = |\vec{a}|^2 |\vec{b}|^2}$$

$$\Leftrightarrow |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$