1/31 - Additional Problems

① Find an equation for the plane consisting of all points that are equidistant from the points $(1,0,-2)$ and $(3,4,0)$.

Solution

The first thing to notice is that this plane must be perpendicular to the line joining $(1,0,-2)$ and $(3,4,0)$. This means the normal vector to the plane is parallel to the line connecting the two points, that is:

$$\vec{n} = (3,4,0) - (1,0,-2) = \langle 2,4,2 \rangle$$

All we need now is a point on the plane. The midpoint of $(1,0,-2)$ and $(3,4,0)$ is, by definition, equidistant from the two points, and is thus on the plane. The midpoint is:

$$\frac{(1,0,-2)+(3,4,0)}{2} = \frac{(4,4,-2)}{2} = (2,2,-1) = P$$

So, an equation for the plane is:

$$\vec{n} \cdot [(x,y,z) - P] = 0$$
$$\langle 2,4,2 \rangle \cdot \langle x-2, y-2, z+1 \rangle = 0$$
$$2(x-2) + 4(y-2) + 2(z+1) = 0$$
$$\boxed{\begin{array}{c} 2x + 4y + 2z = 10 \\ \text{-or- } \; x + 2y + z = 5 \end{array}}$$

② Two particles travel along the space curves $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\vec{r}_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle$. Do the particles collide? Do their paths intersect?

Solution

To collide, they have to be in the same place, at the same time, i.e., the equation $\vec{r}_1(t) = \vec{r}_2(t)$ has a solution. If this is true, then

$$\begin{cases} t = 1+2t & ① \\ t^2 = 1+6t & ② \\ t^3 = 1+14t & ③ \end{cases}$$

is solvable. ① $\Rightarrow t = -1$. Plugging this into ② we get $(-1)^2 = 1 = 1+6(-1) = -5$, which is false, so the particles do not collide. On the other hand, for their paths cross, then $\vec{r}_1(t) = \vec{r}_2(s)$ for some t and s. This means

$$\begin{cases} t = 1+2s & ① \\ t^2 = 1+6s & ② \\ t^3 = 1+14s & ③ \end{cases}$$

is solvable

Plugging ① into ② we have

$$(1+2s)^2 = 1+4s+4s^2 = 1+6s \Rightarrow 4s^2 - 2s = 2s(2s-1) = 0$$
$$\Rightarrow s = 0, \tfrac{1}{2}$$

If $s=0$, then $t=1$. Plugging this into ③ we get $(1)^3 = 1 = 1+14(0) = 1$, so the paths do cross.

③ Let $\vec{r}$ be a vector function, and suppose $\vec{r}\,''$ exists. Show $\frac{d}{dt}\left[\vec{r}(t)\times\vec{r}\,'(t)\right]=\vec{r}(t)\times\vec{r}\,''(t)$.

Solution

The derivative of a cross product satisfies the product rule, so since $\vec{a}\times\vec{a}=\vec{0}$ for any vector $\vec{a}$, we have:

$$\frac{d}{dt}\left[\vec{r}(t)\times\vec{r}\,'(t)\right]=\vec{r}\,'(t)\times\vec{r}\,'(t)+\vec{r}(t)\times\vec{r}\,''(t)$$

$$=\vec{0}+\vec{r}\,'(t)\times\vec{r}\,''(t)=\vec{r}(t)\times\vec{r}\,''(t)$$

④ If $\vec{r}(t)\neq\vec{0}$, show $\frac{d}{dt}|\vec{r}(t)|=\frac{1}{|\vec{r}(t)|}\vec{r}(t)\cdot\vec{r}\,'(t)$

Hint: $|\vec{r}(t)|^2=\vec{r}(t)\cdot\vec{r}(t)$

Solution

The derivative of a dot product satisfies the product rule, so:

$$\frac{d}{dt}|\vec{r}(t)|=\frac{d}{dt}\left(\vec{r}(t)\cdot\vec{r}(t)\right)^{1/2}=\frac{1}{2}\left(\vec{r}\,'(t)\cdot\vec{r}(t)+\vec{r}(t)\cdot\vec{r}\,'(t)\right)\left(\vec{r}(t)\cdot\vec{r}(t)\right)^{-1/2}$$

$$=\frac{\vec{r}(t)\cdot\vec{r}\,'(t)}{\sqrt{\vec{r}(t)\cdot\vec{r}(t)}}=\frac{1}{|\vec{r}(t)|}\vec{r}(t)\cdot\vec{r}\,'(t)$$

⑤ If $\vec{u}(t) = \vec{r}(t) \cdot [\vec{r}\,'(t) \times \vec{r}\,''(t)]$, show

$\vec{u}\,'(t) = \vec{r}(t) \cdot [\vec{r}\,'(t) \times \vec{r}\,'''(t)]$

Solution (Remember $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$.)

This is just the product rule:

$$\begin{aligned}
\vec{u}\,'(t) &= \underbrace{\vec{r}\,'(t) \cdot [\vec{r}\,'(t) \times \vec{r}\,''(t)]}_{\vec{0}} + \vec{r}(t) \cdot [\vec{r}\,'(t) \times \vec{r}\,''(t)]' \\
&= \vec{0} + \vec{r}(t) \cdot [\vec{r}\,''(t) \times \vec{r}\,''(t) + \vec{r}\,'(t) \times \vec{r}\,'''(t)] \\
&= \vec{r}(t) \cdot [\vec{0} + \vec{r}\,'(t) \times \vec{r}\,'''(t)] \\
&= \vec{r}(t) \cdot [\vec{r}\,'(t) \times \vec{r}\,'''(t)].
\end{aligned}$$