2/7 - Extra Problems

① The <u>Frenet-Serret formulas</u> are

$$\frac{d\vec{T}}{ds} = k\vec{N}, \quad \frac{d\vec{N}}{ds} = -k\vec{T} + \tau\vec{B}, \quad \frac{d\vec{B}}{ds} = -\tau\vec{N}.$$

Show the formula for $\frac{d\vec{N}}{ds}$ by using the other two, and the fact that $\vec{N} = \vec{B} \times \vec{T}$.

<u>Hint</u>: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

<u>Solution</u>:

$$\frac{d\vec{N}}{ds} = \left(\frac{d\vec{B}}{ds} \times \vec{T}\right) + \left(\vec{B} \times \frac{d\vec{T}}{ds}\right) = \left(-\tau\vec{N} \times \vec{T}\right) + \left(\vec{B} \times k\vec{N}\right)$$

$$= -\tau\left[(\vec{B} \times \vec{T}) \times \vec{T}\right] + k\left[\vec{B} \times (\vec{B} \times \vec{T})\right]$$

$$= \tau\left[\vec{T} \times (\vec{B} \times \vec{T})\right] + k\left[\vec{B} \times (\vec{B} \times \vec{T})\right] \qquad (\vec{a} \times \vec{b} = -\vec{b} \times \vec{a})$$

$$= \tau\left((\vec{T} \cdot \vec{T})\vec{B} - (\vec{T} \cdot \vec{B})\vec{T}\right) + k\left((\vec{B} \cdot \vec{T})\vec{B} - (\vec{B} \cdot \vec{B})\vec{T}\right)$$

$$= \tau(\vec{B} - 0\vec{T}) + k(0\vec{B} - \vec{T}) = \tau\vec{B} - k\vec{T}.$$

② If a particle with mass m moves with position vector $\vec{r}(t)$, then its <u>angular momentum</u> is defined as $\vec{L}(t) = m\vec{r}(t) \times \vec{v}(t)$ and its <u>torque</u> as $\vec{\tau}(t) = m\vec{r}(t) \times \vec{a}(t)$. Show that $\vec{L}'(t) = \vec{\tau}(t)$. Deduce that if $\vec{\tau}(t) = 0$ for all t, then $\vec{L}(t)$ is constant.
(This is the <u>Law of Conservation of Angular Momentum.</u>)

<u>Solution</u>

Recall $\vec{v}(t) = \vec{r}\,'(t)$ and $\vec{a}(t) = \vec{r}\,''(t)$.
Let's write $\vec{L}(t) = m\vec{r}(t) \times \vec{r}\,'(t)$. Then

$$\vec{L}'(t) = m\left[\vec{r}\,'(t) \times \vec{r}\,'(t) + \vec{r}(t) \times \vec{r}\,''(t)\right]$$
$$= m\vec{r}(t) \times \vec{r}\,''(t) = m\vec{r}(t) \times \vec{a}(t) = \vec{\tau}(t)$$

If $\vec{\tau}(t) = 0$ for all t, then $\vec{L}'(t) = \vec{0}$.
If $\vec{L}(t) = \langle f(t), g(t), h(t)\rangle$, this means $f'(t) = 0$, $g'(t) = 0$, and $h'(t) = 0$, so that $f(t) = c_1$, $g(t) = c_2$, and $h(t) = c_3$.
Thus $\vec{L}(t) = \langle c_1, c_2, c_3\rangle$, a constant function.