Solutions
Math 220
HW # 9
November 14, 2018

**Exercise 1.** *A function* $f : \mathbb{Z} \to \mathbb{Z}$ *is defined by* $f(n) = 2n + 1$*. Determine whether* $f$ *is (a) injective, (b) surjective. Give proof or a counterexample for your claims.*

*Proof.* $f$ is injective, but not surjective. To see that $f$ is not surjective, observe that $2n + 1$ is always odd for integers $n$, so 8, for example, is not in the range of $f$. To see that $f$ is injective, suppose $f(m) = f(n)$. Then

$$\begin{aligned} & f(m) = f(n) \\ \implies & 2m + 1 = 2n + 1 \\ \implies & 2m = 2n \\ \implies & m = n \end{aligned}$$

□

**Exercise 2.** *A function* $f : \mathbb{Z} \to \mathbb{Z}$ *is defined by* $f(n) = n - 3$*. Determine whether* $f$ *is (a) injective, (b) surjective. Give proof or a counterexample for your claims.*

*Proof.* $f$ is a bijection. To see that $f$ is surjective, given an integer $n \in \mathbb{Z}$, observe that $f(n + 3) = (n + 3) - 3 = n$. To see that $f$ is injective, suppose $f(m) = f(n)$. Then

$$\begin{aligned} & f(m) = f(n) \\ \implies & m - 3 = n - 3 \\ \implies & m = n \end{aligned}$$

□

**Exercise 3.** *How many integers must you pick in order to be sure that at least two of them have the same remainder when divided by* 15*?*

*Proof.* When dividing by 15, there are 15 possible remainders: 0, 1,..., 14. So if we choose 16 integers, because there are only 15 possible choices for remainders, at least two of them must have the same remainder by the Pigeonhole Principle. □

**Exercise 4.** *Let* $X = \{1, 2, 3, 4\}$ *and consider the functions* $f, g, h : X \to X$ *given by*

$$\begin{aligned} f &= \{(1, 2), (2, 1), (3, 1), (4, 4)\} \\ g &= \{(1, 2), (2, 2), (3, 1), (4, 3)\} \\ h &= \{(1, 1), (2, 3), (3, 1), (4, 3)\} \end{aligned}$$

(*a*) *Find the following compositions:* $f \circ g$*,* $h \circ f$*, and* $g^2 = g \circ g$*.*

*(b) Find the compositions:* $h \circ g \circ f$ *and* $f \circ g \circ h$.

*Solution.*

(a)
- $f \circ g = \{(1,1),(2,1),(3,2),(4,1)\}$
- $h \circ f = \{(1,3),(2,1),(3,1),(4,3)\}$
- $g \circ g = \{(1,2),(2,2),(3,2),(4,1)\}$

(b)
- $h \circ g \circ f = \{(1,3),(2,3),(3,3),(4,1)\}$
- $f \circ g \circ h = \{(1,1),(2,2),(3,1),(4,2)\}$

□

**Exercise 5.** *Consider the functions* $f, g : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ *given by* $f(m,n) = (3m-4n, 2m+n)$ *and* $g(m,n) = (5m+n, m)$. *Find formulas for* $f \circ g$ *and* $g \circ f$.

*Solution.*

$$\begin{aligned}(f \circ g)(m,n) &= f(5m+n, m)\\ &= (3(5m+n) - 4(m), 2(5m+n) + (m))\\ &= (11m - 3n, 11m + 2n)\\ (g \circ f)(m,n) &= g(3m-4n, 2m+n)\\ &= (5(3m-4n) + (2m+n), 2m+n)\\ &= (17m - 19n, 2m+n)\end{aligned}$$

□

**Exercise 6.** *Let* $X, Y, Z$ *be sets and suppose* $f : X \to Y$ *and* $g : Y \to Z$. *Suppose that* $g \circ f : X \to Z$ *is a bijection. Prove that* $f$ *is an injection and* $g$ *is a surjection.*

*Proof.*

$f$ is injective: Suppose $f(x_1) = f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$, which is equivalent to $(g \circ f)(x_1) = (g \circ f)(x_2)$. But since $g \circ f$ is injective, we have $x_1 = x_2$. Theorefore $f$ is injective.

$g$ is surjective: Let $z \in Z$. Since $g \circ f$ is surjective, there is an $x \in X$ such that $(g \circ f)(x) = z$. Set $y = f(x)$. Then $g(y) = g(f(x)) = (g \circ f)(x) = z$. Thus $g$ is surjective.

□

**Exercise 7.** *Let* $X$ *and* $Y$ *be sets and let* $A \subset X$. *Is* $A = f^{-1}(f(A))$? *If so, give a proof, if not, give a counter example and prove whichever of* $A \subset f^{-1}(f(A))$ *or* $A \supset f^{-1}(f(A))$ *are true.*

*Proof.*

$$\begin{aligned} & x \in A \\ \implies & f(x) \in f(A) \\ \implies & x \in f^{-1}(f(A)) \end{aligned}$$

So $A \subset f^{-1}(f(A))$, however the other direction is not true. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Let $A = [0, \infty)$. Then $f(A) = [0, \infty)$, however $f^{-1}(f(A)) = (-\infty, \infty) \neq A$. □

**Exercise 8.** *Prove that the function* $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{5\}$ *given by* $f(x) = \dfrac{5x+1}{x-2}$ *is bijective. What is the inverse function* $f^{-1} : \mathbb{R} \setminus \{5\} \to \mathbb{R} \setminus \{2\}$*?*

*Proof.* Let $x, y \in \mathbb{R} \setminus \{2\}$ and suppose $f(x) = f(y)$. Then

$$\begin{aligned} & f(x) = f(y) \\ \implies & \frac{5x+1}{x-2} = \frac{5y+1}{y-2} \\ \implies & (5x+1)(y-2) = (5y+1)(x-2) \\ \implies & 5xy + y - 10x - 2 = 5yx + x - 10y - 2 \\ \implies & y - 10x = x - 10y \\ \implies & 11y = 11x \\ \implies & x = y \end{aligned}$$

Therefore $f$ is injective.
Now let $y \in \mathbb{R} \setminus \{5\}$. Set $f(x) = y$ and try to solve for $y$:

$$\begin{aligned} & y = \frac{5x+1}{x-2} \\ \implies & y(x-2) = 5x+1 \\ \implies & yx - 2y = 5x + 1 \\ \implies & yx - 5x = 2y + 1 \\ \implies & x = \frac{2y+1}{y-5} \end{aligned}$$

Since $\dfrac{2y+1}{y-5} \in \mathbb{R} \setminus \{2\}$ and $f\left(\dfrac{2y+1}{y-5}\right) = y$, we have that $f$ is surjective. Observing the computation to show $f$ is surjective, we see the inverse must be

$$f^{-1}(x) = \frac{2x+1}{x-5}.$$

□

**Exercise 9.** *Recall that for an integer* $c \in \mathbb{Z}$*, the set* $c\mathbb{Z} = \{cn | n \in \mathbb{Z}\}$*. Show that* $3\mathbb{Z}$ *and* $5\mathbb{Z}$ *have the same cardinality.*

*Proof.* There are bijections $f : \mathbb{Z} \to 3\mathbb{Z}$, $x \mapsto 3x$, and $g : \mathbb{Z} \to 5\mathbb{Z}$ , $x \mapsto 5x$, as described in class. Since $f$ is a bijection, $f^{-1} : 3\mathbb{Z} \to \mathbb{Z}$ is a bijection as well. Therefore $g \circ f : 3\mathbb{Z} \to 5\mathbb{Z}$ is a bijection, therefore $3\mathbb{Z}$ and $5\mathbb{Z}$ have the same cardinality. □

**Exercise 10.** *Prove that the rational numbers, $\mathbb{Q}$, are countable. (You may use the fact that the union of two countable sets is countable.)*

*Proof.* Recall from class that $\mathbb{Q}^+$ is countable. The function $f : \mathbb{Q}^+ \to \mathbb{Q}^-$ given by $f(x) = -x$ is a bijection between the two sets, so $\mathbb{Q}^-$ is countable as well. Since $\mathbb{Q} = \mathbb{Q}^- \cup \mathbb{Q}^+ \cup \{0\}$, it is the union of two countably infinite sets and a finite set, and therefore must be countable as well. □