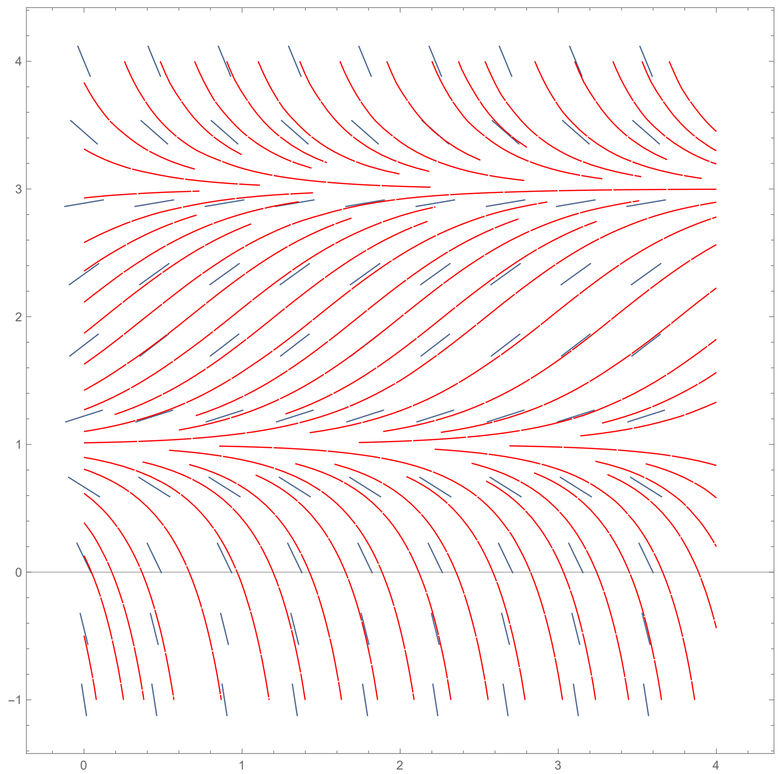
Math 307 - Differential Equations - Spring 2017

Exam 1 Solutions

Problem 1. *For parts (a)-(c), we will choose* $a = 4$ *and* $b = 1$.

(a) *In this case we need* $q < \dfrac{a^2}{4b} = \dfrac{16}{4} = 4$ *so choose* $q = 3$. *Observe the direction field with several integral curves plotted*

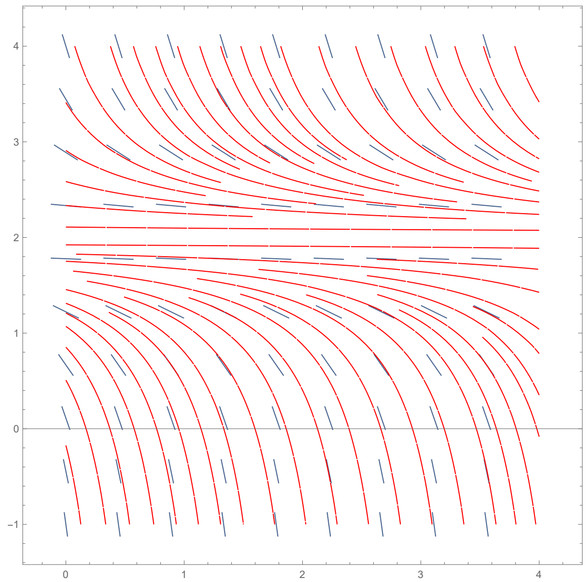


There is some peculiar behavior around $y = 1$ *and* $y = 3$. *Remember that looking at the limit as* $t \to \infty$ *we are following the solutions as we go to the right. It looks* $y_1 = 1$ *and* $y_2 = 3$. *It looks like there are constant solutions at these two values, so let's check. The differential equation here is*

$$y' = -y^2 + 4y - 3 = -(y-1)(y-3)$$

so we can see that $y = 1$ *and* $y = 3$ *are indeed constant solutions. This verifies that* $y_1 = 1$ *and* $y_2 = 3$.

(b) *This time we must choose* $q = 4$. *The direction field with integral curves looks like this now*

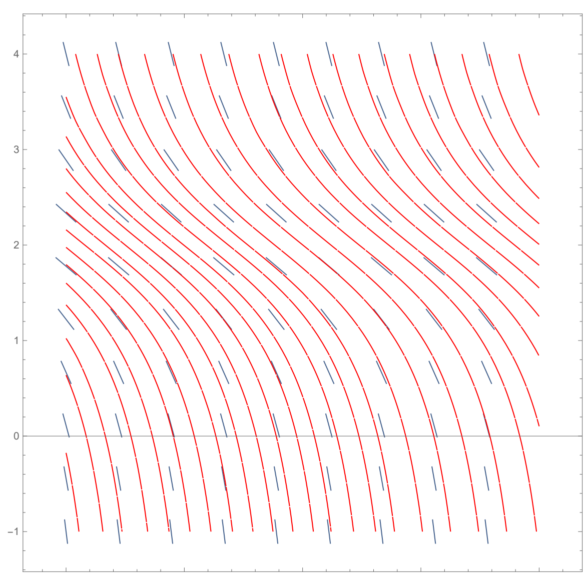


and this time we see interesting behavior at $y = 2$ which suggests that $y_1 = 2$. It again appears that there is a constant solution there, so let's look at the differential equation to look for constant solutions

$$y' = -y^2 + 4y - 4 = -(y-2)^2.$$

Thus we see that the constant solution is just $y = 2$ which verifies that $y_1 = 2$.

(c) *Here, we need $q > 4$, so choose $q = 5$. The integrals curves in this case appear as*



We can see that they all go to $-\infty$.

Problem 2.

(a) *Solve the differential equation by separation:*

$$I' = rI(S-I) \quad \Longrightarrow \quad \frac{dI}{I(S-I)} = r\ dt$$

We can see that if $I = 0$ or $I = S$, we have constant solutions, so assume $I \neq 0, S$. Then, integrate both sides:

$$\int \frac{dI}{I(S-I)} = \int \left(\frac{1/S}{I} + \frac{1/S}{S-I}\right) dI = \frac{1}{S}\left(\ln I - \ln(S-I)\right) = \frac{1}{S}\ln\frac{I}{S-I}$$

and

$$\int r\ dt = rt + C$$

so

$$\frac{1}{S}\ln\frac{I}{S-I} = rt + C.$$

Multiply by S then exponentiate to get

$$\frac{I}{S-I} = Ce^{rSt}.$$

Plug in the initial value $I(0) = I_0$ to get

$$\frac{I_0}{S-I_0} = Ce^0 = C$$

Solve for I in the solution above and plug in the value for C to get

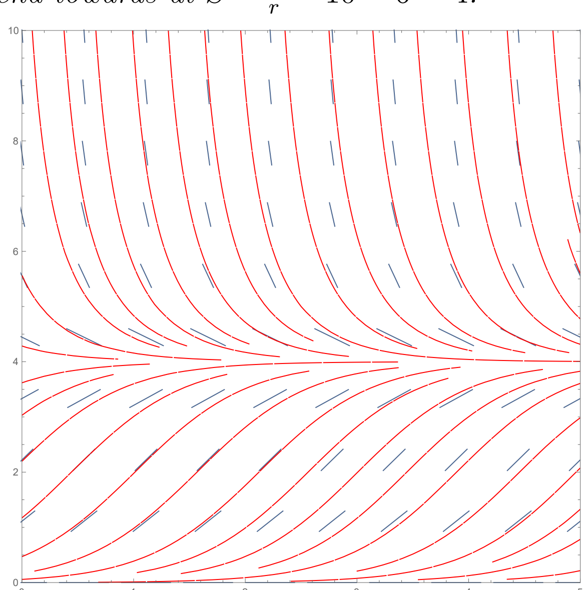
$$I = \frac{SI_0}{(S-I_0)e^{-rSt} + I_0}.$$

To see what happens to the population ($I(t)$) as time goes on, take the limit as $t \to \infty$:

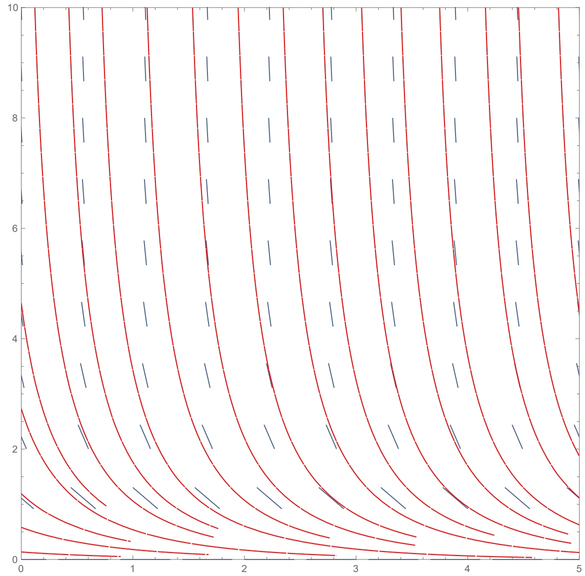
$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty}\frac{SI_0}{(S-I_0)e^{-rSt} + I_0}\frac{SI_0}{(S-I_0)(0) + I_0} = \frac{SI_0}{I_0} = S.$$

This means the whole population gets infected!

(b) *If the vaccination rate is high enough, then the disease is eradicated, otherwise the number of zombies stabilizes at $S - \frac{q}{r}$. In the graphics below, $S = 10$ and $r = 0.5$. In this first one $q = 3$ ($q < rS$). We can see that there is an equilibrium that the number of infected tend towards at $S - \frac{q}{r} = 10 - 6 = 4$.*



In this next one $q = 6$ ($q \geq rS$). We can see that in all cases, the number of infected tends to zero.



(c) *The differential equation is*

$$I' = rI(S - I) - qI$$

which we can rewrite as

$$I' + (q - rS)I = -rI^2.$$

This is a Bernoulli equation with $n = 2$, so make the substitution $u = I^{1-n}$ which will turn the differential equation into

$$u' + (rS - q)u = r.$$

Solving this using the integrating factor

$$\mu = e^{\int^t (rS-q)dx} = e^{(rS-q)t}$$

we get the solution

$$u = \frac{r}{rS - q} + Ce^{(q-rS)t}$$

and plugging back in $u = I^{-1}$ we can solve for I to get

$$I = \frac{S - \frac{q}{r}}{1 + Ce^{(q-rS)t}}.$$

Using the initial condition $I(0) = I_0$, one can find that $C = \frac{r(S-I_0)-q}{I_0(rS-q)}$.

Now we just need to take the limits in the three different cases:

($q < rS$) *In this case, $e^{(q-rS)t} \to 0$ as $t \to \infty$, so we get*

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \frac{S - \frac{q}{r}}{1 + Ce^{(q-rS)t}} = \frac{S - \frac{q}{r}}{1 + C(0)} = S - \frac{q}{r}.$$

($q = rS$) *If $q = rS$ the differential equation becomes $I' = -rI^2$ which has the solution $I = \frac{1}{rt+C}$. We can see that*

$$\lim_{t\to\infty} I(t) = 0.$$

($q > rS$) If $q > rS$, then $e^{(q-rS)t} \to \infty$ as $t \to \infty$. Thus

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \frac{S - \frac{q}{r}}{1 + Ce^{(q-rS)t}} = 0.$$

Problem 3. *Make the substitution $u = \arctan y$. Then $u' = \frac{1}{1+y^2}y'$ and plugging this into the differential equation gives*

$$u' + \frac{2}{x}u = \frac{2}{x}.$$

This is now a linear equation, so use an integrating factor:

$$\mu = e^{\int^x \frac{2}{s} ds} = e^{2\ln|x|} = x^2.$$

Then we get the solution of the DE in u:

$$u = x^{-2} \int (x^2)\left(\frac{2}{x}\right) dx = x^{-2} \int 2x\ dx = x^{-2}\left(x^2 + C\right) = 1 + Cx^{-2}.$$

Since $u = \arctan y$ we have

$$\arctan y = 1 + Cx^{-2} \quad \Longrightarrow \quad y = \tan\left(1 + Cx^{-2}\right).$$

Problem 4.

(a) *y_1 being a solutions means $y_1' = p(x)y_1^2 + q(x)y_1 + r(x)$. We just need to plug in $y = y_1 + \dfrac{1}{u}$ as the problem suggests. $y' = y_1' - \dfrac{1}{u^2}u'$, so plugging in gives*

$$\begin{aligned}
y_1' - \frac{1}{u^2}u' &= p(x)\left(y_1 + \frac{1}{u}\right)^2 + q(x)\left(y_1 + \frac{1}{u}\right) + r(x) \\
y_1' - \frac{1}{u^2}u' &= p(x)\left(y_1^2 + 2\frac{y_1}{u} + \frac{1}{u^2}\right) + q(x)\left(y_1 + \frac{1}{u}\right) + r(x) \\
y_1' - \frac{1}{u^2}u' &= \left(p(x)y_1^2 + q(x)y_1 + r(x)\right) + p(x)\left(2\frac{y_1}{u} + \frac{1}{u^2}\right) + q(x)\frac{1}{u} \\
\cancel{y_1'} - \frac{1}{u^2}u' &= \cancel{y_1'} + p(x)\left(2\frac{y_1}{u} + \frac{1}{u^2}\right) + q(x)\frac{1}{u} \\
-\frac{1}{u^2}u' &= p(x)\left(2\frac{y_1}{u} + \frac{1}{u^2}\right) + q(x)\frac{1}{u} \\
u' &= p(x)\left(-2y_1u - 1\right) - q(x)u \\
u' + \left(2y_1p(x) + q(x)\right)u &= -p(x)
\end{aligned}$$

Which is a linear differential equation in u.

(b) *Matching up the differential equation with the general form, we get that $p(x) = 1, q(x) = 2x, r(x) = x^2 - 1$. Plug these, along with $y_1 = -x$ into the equation we found in part (a) to get*

$$\begin{aligned}
u' + (2(-x)(1) + 2x)u &= -1 \\
u' + 0u &= -1 \\
u' &= -1
\end{aligned}$$

Thus $u = -x + C$ *and plugging this into* $y = y_1 + \frac{1}{u}$ *to get* y *we have*

$$y = -x + \frac{1}{-x + C}.$$

Problem 5. *We have to solve this in two pieces:*

$\underline{0 \le x \le 1}$ *Here,* $g(x) = 1$ *and so the differential equation is*

$$y' + 2y = 1.$$

This is a linear differential equation, so

$$\mu = e^{\int^x 2ds} = e^{2x}$$

and

$$y = e^{-2x}\left(\int \left(e^{2x}\right)(1)dx\right) = e^{-2x}\left(\frac{1}{2}e^{2x} + C\right) = \frac{1}{2} + Ce^{-2x}.$$

Using the initial value, we get

$$y(0) = \frac{1}{2} + Ce^0 = \frac{1}{2} + C = 0 \quad \Longrightarrow \quad C = -\frac{1}{2}.$$

Thus

$$y = \frac{1}{2} - \frac{1}{2}e^{-2x} = \frac{1}{2}\left(1 - e^{-2x}\right).$$

$\underline{x > 1}$ *Here* $g(x) = 0$ *and so the differential equation is*

$$y' + 2y = 0$$

which gives as its solution

$$y = Ce^{-2x}.$$

The initial value CANNOT *be used here since the initial value is at* $x = 0$*, but* $x > 1$ *here. To find* C*, we match up this solution with the solution to the previous part at* $x = 1$*, the* x*-value where they meet up. From the first solution*

$$y(1) = \frac{1}{2}\left(1 - e^{-2}\right)$$

and the new one

$$y = Ce^{-2}.$$

Setting these equal to each other, we get

$$Ce^{-2} = \frac{1}{2}\left(1 - e^{-2}\right) \quad \Longrightarrow \quad C = \frac{1}{2}(1 - e^{-2})e^2 = \frac{1}{2}\left(e^2 - 1\right).$$

Thus the solution for $x > 1$ *is*

$$y = \frac{1}{2}(e^2 - 1)e^{-2x}.$$

Putting the two pieces together, we have the solution

$$y = \begin{cases} \frac{1}{2}\left(1 - e^{-2x}\right), & 0 \le x \le 1 \\ \frac{1}{2}(e^2 - 1)e^{-2x}, & x > 1 \end{cases}.$$

Problem 6. *First recall the Mean Value Theorem*

Theorem (Mean Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a,b]$ and differentiable on (a,b), then*

$$f(a) - f(b) = f'(c)(a-b)$$

for some $c \in (a,b)$.

(a) *Holding x constant allows us to think of $f(x,y)$ as a function of y only. Let's reinforce this by writing $g_x(y) = f(x,y)$. Then, applying the Mean Value Theorem to g_x, we have*

$$g_x(y_1) - g_x(y_2) = g_x'(c)(y_1 - y_2).$$

Replacing g_x with f we have

$$f(x,y_1) - f(x,y_2) = \frac{\partial f}{\partial y}(x,c)(y_1 - y_2).$$

Now, take the absolute value of both sides

$$|f(x,y_1) - f(x,y_2)| = \left|\frac{\partial f}{\partial y}(x,c)(y_1 - y_2)\right| = \left|\frac{\partial f}{\partial y}(x,c)\right| |y_1 - y_2|.$$

Now, let K be the maximum value of $\frac{\partial f}{\partial y}$ on the rectangle D, then

$$|f(x,y_1) - f(x,y_2)| = \left|\frac{\partial f}{\partial y}(x,c)\right| |y_1 - y_2| \leq K|y_1 - y_2|$$

as desired.

(b) *Let $\varphi(x)$ and $\psi(x)$ be solutions of (5), then*

$$\varphi(x) = \int_0^x f(s,\varphi(s))\ ds \quad \text{and} \quad \psi(x) = \int_0^x f(s,\psi(s))\ ds.$$

Take the difference of $\varphi(x)$ and $\psi(x)$ to get

$$\varphi(x) - \psi(x) = \int_0^x f(s,\varphi(s))\ ds - \int_0^x f(s,\psi(s))\ ds = \int_0^x [f(s,\varphi(s)) - f(s,\psi(s))]\, ds$$

as desired.

(c) *Recall the fact that $\left|\int f\ dx\right| \leq \int |f| dx$, then applying absolute value to the equation in part (b) we get*

$$|\varphi(x) - \psi(x)| = \left|\int_0^x [f(s,\varphi(s)) - f(s,\psi(s))]\, ds\right| \leq \int_0^x |f(s,\varphi(s)) - f(s,\psi(s))|\, ds$$

as desired.

(d) *Combine the inequality we found in part (a) with the result of part (c). Use $y_1 = \varphi(x)$ and $y_2 = \psi(x)$.*

$$\begin{aligned} |\varphi(x) - \psi(x)| &\leq \int_0^x |f(s,\varphi(s)) - f(s,\psi(s))|\, ds \\ &\leq \int_0^x K\, |\varphi(s) - \psi(s)|\, ds = K \int_0^x |\varphi(s) - \psi(s)|\, ds \end{aligned}$$

as desired.

(e) *Letting* $U(x) = \int_0^x |\varphi(s) - \psi(s)|\, ds$ *we have* $U'(x) = |\varphi(x) - \psi(x)|$ *and so the inequality from part (d) is*

$$U' \leq KU.$$

Rearranging we get

$$U' - KU \leq 0$$

which looks like a linear differential equation. The integrating factor would be

$$\mu(x) = e^{\int^x -K ds} = e^{-Kx}$$

so multiplying the equation by this gives

$$e^{-Kx}U' - Ke^{-Kx}U = \left(e^{-Kx}U\right)' \leq 0.$$

Integrating both sides from 0 *to* x *gives*

$$e^{-Kx}U \leq 0$$

and since $e^{-Kx} > 0$ *we can divide by it to get*

$$U \leq 0.$$

Thus, combining this with the original inequality, we have

$$U' \leq KU \leq 0$$

so that

$$U' \leq 0.$$

Since $U'(x) = |\varphi(x) - \psi(x)| \geq 0$ *we therefore have the desired conclusion:*

$$U'(x) \equiv 0.$$

(f) *Continuing from part (e),*

$$U'(x) = |\varphi(x) - \psi(x)| = 0$$

implies that

$$\varphi(x) - \psi(x) = 0 \quad \Longrightarrow \quad \varphi(x) = \psi(x)$$

which gives the uniqueness, as desired.