Math 10B - Calculus of Several Variables II - Spring 2011
March 27, 2011
Practice Final

Name: ______________________________

There are 160 possible points on this exam. A perfect score will be 130 points. There is no need to use calculators on this exam. All electronic devices should be turned off and put away. The only things you are allowed to have are: a writing utensil(s) (pencil preferred), an eraser, and an exam. All answers should be given as exact, closed form numbers as opposed to decimal approximations (i.e. π as opposed to 3.14159265358979...). Cheating is strictly forbidden. You may leave when you are done. Good luck!

Problem	Score
1	/10
2	/10
3	/30
4	/30
5	/20
6	/20
7	/20
8	/10
9	/10
Score	/130

Problem 1 (10 points). *The average value of a function $f(x,y)$ over a region R is the value*

$$AV(f,R) = \frac{1}{Area(R)} \iint_R f(x,y)\, dA.$$

Find the average value of $f(x,y) = x \sin xy$ over the region $R = [0,1] \times [0,1]$.

$$Area(R) = 1$$

$$AV(f,R) = \frac{1}{1} \int_0^1 \int_0^1 x \sin xy \, dy\, dx$$

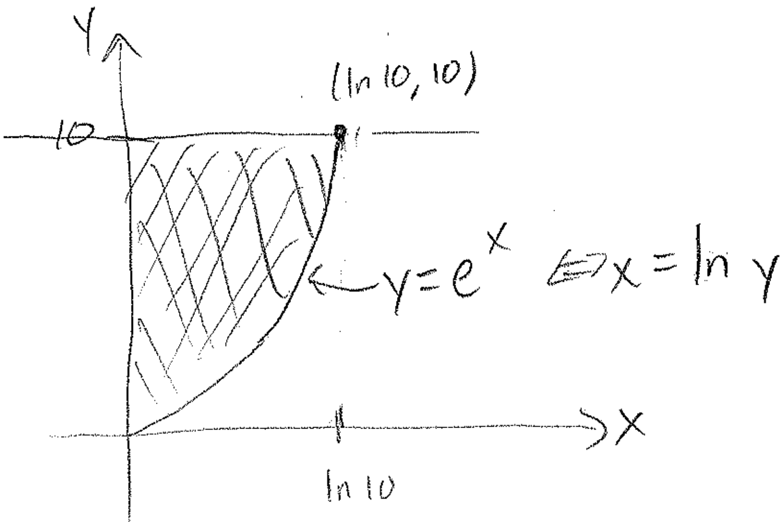
$$= \int_0^1 \frac{\cancel{x} \sin xy}{\cancel{x}} \Big|_0^1 dx = \int_0^1 (\sin x - \sin 0)\, dx$$

$$= \int_0^1 \sin x \, dx = -\cos x \Big|_0^1 = -\cos 1 + \cos 0 = \boxed{1 - \cos 1}$$

Problem 2 (10 points). *Compute the integral*

$$\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\,dydx$$

and draw the region of integration.



$$\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\,dy\,dx = \int_0^{10}\int_0^{\ln y} \frac{1}{\ln y}\,dx\,dy$$

$$= \int_0^{10} \left.\frac{x}{\ln y}\right|_0^{\ln y} dy = \int_0^{10} (1-0)\,dy = \boxed{10}$$

Problem 3 (30 points).

(a) *(10 points) Show that* $\frac{\partial(x,y)}{\partial(r,\theta)} = r$ *where* (r,θ) *is polar coordinates.*

(b) *(10 points) Combine*

$$\overset{①}{\int_0^2 \int_0^x \sqrt{x^2+y^2}\,dy\,dx} + \overset{②}{\int_2^{2\sqrt{2}} \int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\,dy\,dx}.$$

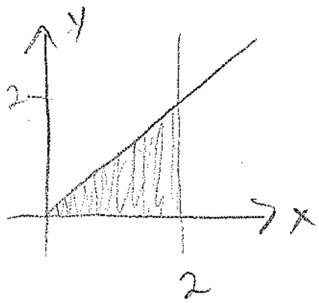
(Hint: Use polar coordinates.)

(c) *(10 points) Compute the integral in part (b).*

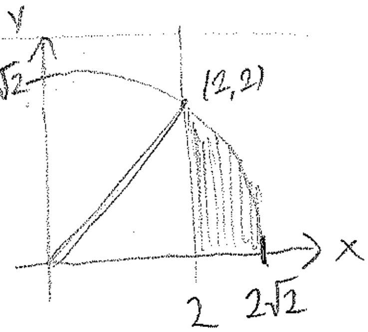
ⓐ $x = r\cos\theta,\ y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \boxed{r}$$

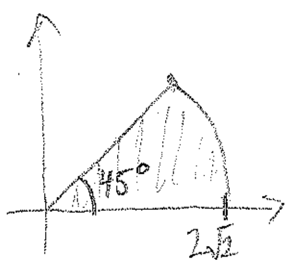
ⓑ Region ①:



Region ②



Together:



$$\int_0^2\int_0^x \sqrt{x^2+y^2}\,dy\,dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/4}\int_0^{2\sqrt{2}} r\,(r\,dr\,d\theta).$$

ⓒ $$\int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta = \frac{1}{3}\int_0^{\pi/4} (r^3)\Big|_0^{2\sqrt{2}} d\theta = \frac{1}{3}\int_0^{\pi/4} 16\sqrt{2}\,d\theta = \boxed{\frac{4\sqrt{2}\,\pi}{3}}$$

Problem 4 (30 points).

(a) *(10 points) State the equations to convert Cartesian coordinates to cylindrical coordinates and give the Jacobian of this transformation.*

(b) *(10 points) State the equations to convert Cartesian coordinates to spherical coordinates and give the Jacobian of this transformation.*

(c) *(10 points) Show that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dxdydz = 2\pi.$$

(a) $x = r\cos\theta \quad y = r\sin\theta \quad z = z \quad \dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

(b) $x = \rho\cos\theta\sin\varphi \quad y = \rho\sin\theta\sin\varphi \quad z = \rho\cos\varphi, \quad \dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \rho^2\sin\varphi$

(c) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dxdydz$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\rho e^{-\rho^2}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\rho^3 e^{-\rho^2}\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$\begin{pmatrix} u=\rho^2 \\ dv=\rho e^{-\rho^2}d\rho \end{pmatrix}$ $$= \int_0^{\pi}\int_0^{2\pi}\left(\rho^2\left(-\frac{1}{2}e^{-\rho^2}\right)\Big|_0^{\infty} + \int_0^{\infty}\rho e^{-\rho^2}d\rho\right)\sin\varphi\,d\theta\,d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi}\left(\lim_{t\to\infty}\left[\left(\frac{-\rho^2}{2e^{\rho^2}}\right) - \frac{1}{2e^{\rho^2}}\right]_0^{t}\right)\sin\varphi\,d\theta\,d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi}\left(\lim_{t\to\infty}\left(\frac{-t^2}{2e^{t^2}} - \frac{1}{2e^{t^2}}\right) - \left(0 - \frac{1}{2e^0}\right)\right)\sin\varphi\,d\theta\,d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi}\left(0-0-0+\frac{1}{2}\right)\sin\varphi\,d\theta\,d\varphi = \frac{1}{2}\int_0^{\pi}\int_0^{2\pi}\sin\varphi\,d\theta\,d\varphi$$

$$= \pi\int_0^{\pi}\sin\varphi\,d\varphi = \pi(-\cos\varphi)\Big|_0^{\pi} = \pi(-\cos\pi + \cos 0) = 2\pi$$

Problem 5 (20 points).

(a) *(10 points) Determine whether $\vec{F}(x,y) = (2xy, x^2)$ is conservative.*

(b) *(10 points) Compute $\int_{\mathbf{c}} \vec{F} \cdot d\vec{s}$ where **c** is the portion of $y = x^{17} + 2x^{15} - 2x^{10} + 7x^7 - 3x^5 + \pi$ from $(0,\pi)$ to $(1, 5+\pi)$.*

(a) $\frac{\partial}{\partial y}(2xy) = 2x \quad \frac{\partial}{\partial x}(x^2) = 2x$

$\therefore$ Conservative.

(b) $\vec{F} = \nabla f$. $f = \int 2xy\,dx = x^2 y + \phi(y)$

$f_y = x^2 + \phi'(y) = x^2 \Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = C$

$\Rightarrow f = x^2 y + C \qquad$ (Use $C=0$)

$\int_C \vec{F} \cdot d\vec{s} = f(1, 5+\pi) - f(0,\pi)$

$= 1^2(5+\pi) - 0^2\pi = \boxed{5+\pi}$

Problem 6 (20 points).

(a) *(10 points) State Green's theorem.*

(b) *(10 points) If f and g are differentiable functions and* **c** *is a piecewise smooth, simple, closed path, show:*

$$\int_{\mathbf{c}} f(x)\,dx + g(y)\,dy = 0.$$

ⓐ Let C be a piecewise smooth, simple, closed curve in $\mathbb{R}^2$, oriented positively. If $D = \partial C$ and M and N are differentiable then

$$\oint_C M dx + N dy = \iint_D (N_x - M_y)\, dA.$$

ⓑ By Green's theorem: If $\vec{c} = \partial D$

$$\int_{\vec{c}} f(x)\,dx + g(y)\,dy = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_D (0-0)\, dA = 0$$

Problem 7 (20 points). *Consider the surface* $x^2+y^2=1$, $-1 \le z \le 1$. *Call this surface* S.

(a) *Find the tangent plane at* $(1,0,0)$.

(b) *Compute the integral* $\iint_S \vec{F}\cdot d\vec{S}$ *where* $\vec{F}(x,y,z)=(x,y,0)$.

ⓐ $X(\theta,z) = (\cos\theta, \sin\theta, z)$, $0 \le \theta \le 2\pi$, $-1 \le z \le 1$.

$T_\theta = (-\sin\theta, \cos\theta, 0)$ $\quad T_z = (0,0,1)$

$$N = \begin{vmatrix} \hat{i} & \hat{j} & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \sin\theta, 0)$$

$X(0,0) = (\cos 0, \sin 0, 0) = (1,0,0)$

$N(0,0) = (1,0,0)$ $\quad N\cdot((x,y,z)-(1,0,0)) = (1,0,0)\cdot(x-1,y,z) = 0$

$\Rightarrow$ Tangent plane is $\boxed{x=1}$.

ⓑ $\iint_S \vec{F}\cdot d\vec{S} = \int_{-1}^{1}\int_0^{2\pi} \vec{F}(X(\theta,z))\cdot N(\theta,z)\, d\theta\, dz$

$= \int_{-1}^{1}\int_0^{2\pi} (\cos\theta, \sin\theta, 0)\cdot(\cos\theta,\sin\theta,0)\, d\theta\, dz$

$= \int_{-1}^{1}\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\, d\theta\, dz$

$= \int_{-1}^{1}\int_0^{2\pi} d\theta\, dz = (1-(-1))(2\pi - 0) = \boxed{4\pi}$

Problem 8 (10 points). *Let S be the sphere of radius 2. Orient ~~B~~ S with outward normals. Let $\vec{F}(x, y, z) = (x, y, z)$. Compute the integral*

$$\iint_S \vec{F} \cdot d\vec{S}.$$

These satisfy the Divergence Theorem, so
(Gauss')

if $S = \partial B$, then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_B \operatorname{div} \vec{F} \, dV = \iiint_B (1+1+1) \, dV = 3 \iiint_B dV$$

$$= 3 \operatorname{Vol}(B) = 3 \cdot \left(\frac{4}{3}\pi(2)^2\right) = \boxed{16\pi}$$

Problem 9 (10 points). *Compute* $\iint_S \text{curl}\vec{F}\cdot d\vec{S}$ *where* $\vec{F}(x,y,z)=(x,y,z)$ *and* S *is the surface* $x^2+y^2+z^2=9^2$, $z\geq 0$.

∂S = circle of radius 9 : $x^2+y^2=9^2$, $z=0$

if $c=\partial S$, $c(t)=(9\cos t, 9\sin t, 0)$, $0\leq t\leq 2\pi$

By Stokes' theorem:

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}(c(t))\cdot c'(t)\,dt$$

$$= \int_0^{2\pi} (9\cos t, 9\sin t, 0)\cdot(-9\sin t, 9\cos t, 0)\,dt$$

$$= \int_0^{2\pi} (-81\cos t\sin t + 81\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} 0\,dt = \boxed{0}$$