HW #2: 5.2: 18, 24, Thm 5.5 for 2×2, 8
5.3: 1, 2, 4, 8, 10
6.1: 2, 4, 6

5.2) (18) Suppose $A \sim B$. Then $\exists C$ s.t.
$B = C^{-1}AC$
It suffices to show that $p_A(\lambda) = p_B(\lambda)$.

$$
\begin{aligned}
p_B(\lambda) &= \det(B - \lambda I) = \det(C^{-1}AC - \lambda I) \\
&= \det(C^{-1}AC - \lambda C^{-1}IC) \\
&= \det(C^{-1}(A-\lambda I)C) \\
&= \det(C^{-1})\det(A-\lambda I)\det(C) \\
&= \det(A - \lambda I) = p_A(\lambda)
\end{aligned}
$$

$\square$ ✓

(24) Suppose that a linear combination w of vectors in $\bigcup_{i=1}^m B_i$ is zero. We can write
$w = w_1 + \cdots + w_m = 0$ where $w_i \in B_i$ $\forall i$.
Then each w_i is an eigenvector of A corresponding to λ_i. Now since $\lambda_1, \cdots, \lambda_m$ are distinct by theorem 5.3 we have that each $w_i = 0$. But we can write each w_i as $w_i = c_1 b_{i_1} + \cdots + c_k b_{i_k}$ where $\{b_{i_1}, \cdots, b_{i_k}\} = B_i$. But B_i is a basis for E_{λ_i} and $w_i = 0$ so $c_1 = \cdots = c_k = 0$.

Thus $\bigcup_{i=1}^m B_i$ is a linearly independent set in $\mathbb{R}^n$.

Thm 5.5 for 2×2

Let $A=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, a real symmetric matrix.

$$\det(A-\lambda I) = (a-\lambda)(c-\lambda)-b^2 = \lambda^2-(a+c)\lambda+(ac-b^2)=0$$

$$\Rightarrow \lambda = \frac{a+c \pm \sqrt{(a+c)^2-4(1)(ac-b^2)}}{2}$$

$$= \frac{a+c \pm \sqrt{a^2+2ac+c^2-4ac+b^2}}{2}$$

$$= \frac{a+c \pm \sqrt{a^2-2ac+c^2+b^2}}{2} = \frac{a+c \pm \sqrt{(a-c)^2+b^2}}{2}$$

So clearly both eigenvalues are real since $(a-c)^2+b^2 \geq 0$.

If $(a-c)^2+b^2 \neq 0$ we have two distinct e.vals and hence the geo. mult. of each one matches the alg. mult.

Now assume that $(a-c)^2+b^2=0$. Then

$$(a-c)^2=b^2=0 \Rightarrow \begin{cases} a-c=0 \\ b=0 \end{cases} \Rightarrow \begin{cases} a=c \\ b=0 \end{cases}$$

Then $A=\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ which has e.val. $\lambda=a$

and e.vecs. $v_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $v_2=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\therefore$ diagonalizable. $\square$

(8) $\begin{vmatrix} -4-\lambda & 6 & -12 \\ 3 & -1-\lambda & 6 \\ 3 & -3 & 8-\lambda \end{vmatrix}$

$= (-4-\lambda)(-8-7\lambda+\lambda^2+18) - 6(24-3\lambda-18) - 12(-9+3+3\lambda)$

$= (-4-\lambda)(\lambda^2-7\lambda+10) - 6(-3\lambda+6) - 12(3\lambda-6)$

$= (-4-\lambda)(\lambda-2)(\lambda-5) + 18(\lambda-2) - 36(\lambda-2)$

$= (\lambda-2)(-\lambda^2-4\lambda+5\lambda+20+18-36)$

$= (\lambda-2)(-\lambda^2+\lambda+2) = -(\lambda-2)(\lambda^2-\lambda-2)$

$= -(\lambda-2)(\lambda-2)(\lambda+1) = 0$

$\Rightarrow \lambda_1 = -1, \ \lambda_2 = \lambda_3 = 2.$

$\underline{\lambda_1 = -1}$

$$\begin{pmatrix} -3 & 6 & -12 \\ 3 & 0 & 6 \\ 3 & -3 & 9 \end{pmatrix} \underset{R_3/3}{\overset{R_1/3,\ R_2/3}{\sim}} \begin{pmatrix} -1 & 2 & -4 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix} \underset{R_3-R_1}{\overset{R_1+R_2}{\sim}} \begin{pmatrix} 0 & 2 & -2 \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\overset{R_1+2R_3}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{Let } x_3 = t \Rightarrow V_1 = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2 = \lambda_3 = 2}$

$$\begin{pmatrix} -6 & 6 & -12 \\ 3 & -3 & 6 \\ 3 & -3 & 6 \end{pmatrix} \underset{R_1/6,\ R_2/3}{\overset{R_3-R_2}{\sim}} \begin{pmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_1+R_2}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $x_2 = s$ & $x_3 = r \Rightarrow V_2 = \begin{pmatrix} s-2r \\ s \\ r \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + r\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

Let
$$C = \begin{pmatrix} -2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
Then
$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

5.3) ① $a_0 = 0,\ a_1 = 1,\ a_k = (a_{k-1} + a_{k-2})/2 \quad k \geq 2$

1, 2, 4, 8, 10

ⓐ $$\begin{pmatrix} a_k \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{k-1} \\ a_{k-2} \end{pmatrix} = \begin{pmatrix} \frac{a_{k-1}}{2} + \frac{a_{k-2}}{2} \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a_{k-1} + a_{k-2}) \\ a_{k-1} \end{pmatrix}$$
$$\therefore\ A = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$$

ⓑ $$\det(A - \lambda I) = \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1 & -\lambda \end{vmatrix} = (\tfrac{1}{2} - \lambda)(-\lambda) - \tfrac{1}{2}$$
$$= \lambda^2 - \tfrac{1}{2}\lambda - \tfrac{1}{2} = (\lambda - 1)(\lambda + \tfrac{1}{2}) = 0$$
$$\Rightarrow \lambda = -1/2,\ 1$$

The process is neutrally stable since $|\lambda| \leq 1$.

ⓒ $\underline{\lambda_1 = -1/2}$
$$\begin{pmatrix} 1 & 1/2 \\ 1 & 1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = c\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
$\underline{\lambda_2 = 1}$
$$\begin{pmatrix} -1/2 & 1/2 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = d\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We need to express x as

$$x = d_1 v_1 + d_2 v_2$$

So:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = d_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + d_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} -d_1 + d_2 = 1 \\ 2d_1 + d_2 = 0 \end{cases} \Rightarrow 3d_1 = -1 \Rightarrow d_1 = -1/3 \Rightarrow d_2 = 2/3$$

So:

$$x = \frac{-1}{3}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now (1) is $A^k x = d_1 \lambda_1{}^k v_1 + d_2 \lambda_2{}^k v_2$

$$\Rightarrow A^k x = \frac{-1}{3}\left(\frac{-1}{2}\right)^k \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{2}{3}(1)^k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{(-1)^{k+1}}{3 \cdot 2^k}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix}$$

The sequence starts out: $0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \ldots$

$$A^1 x = \frac{1}{6}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/6 \\ 2/6 \end{pmatrix} + \begin{pmatrix} 4/6 \\ 4/6 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} \checkmark$$

$$A^2 x = \frac{-1}{12}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/12 \\ -2/12 \end{pmatrix} + \begin{pmatrix} 8/12 \\ 8/12 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/2 \end{pmatrix} = \begin{pmatrix} a_3 \\ a_2 \end{pmatrix} \checkmark$$

$$A^3 x = \frac{1}{24}\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/24 \\ 2/24 \end{pmatrix} + \begin{pmatrix} 16/24 \\ 16/24 \end{pmatrix} = \begin{pmatrix} 15/24 \\ 18/24 \end{pmatrix} = \begin{pmatrix} 5/8 \\ 3/4 \end{pmatrix}$$

$$= \begin{pmatrix} a_4 \\ a_3 \end{pmatrix} \checkmark$$

ⓓ Notice that as k becomes larger

$$\frac{(-1)^{k+1}}{3\cdot 2^k} \to 0$$ so that

$$A^k x \approx \frac{2}{3}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2/3\\2/3\end{pmatrix} = \begin{pmatrix}a_{k+1}\\a_k\end{pmatrix}.$$

So for large k

$$a_k \approx 2/3$$

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② $a_k = a_{k-1} - \left(\frac{3}{16}\right)a_{k-2}$, $k \geq 2$, $a_0 = 0$, $a_1 = 1$

ⓐ $$\begin{pmatrix}a_k\\a_{k-1}\end{pmatrix} = \begin{pmatrix}1 & \frac{-3}{16}\\1 & 0\end{pmatrix}\begin{pmatrix}a_{k-1}\\a_{k-2}\end{pmatrix}$$

So $$A = \begin{pmatrix}1 & -3/16\\1 & 0\end{pmatrix}$$

ⓑ $$P_A(\lambda) = \begin{vmatrix}1-\lambda & -3/16\\1 & -\lambda\end{vmatrix} = -\lambda + \lambda^2 + \frac{3}{16} = 0$$

$$\lambda = \frac{1 \pm \sqrt{1 - 3/4}}{2} = \frac{1 \pm \sqrt{1/4}}{2} = \frac{1 \pm \frac{1}{2}}{2} = \frac{3}{4}, \frac{1}{4}$$

$\therefore$ The process is stable since $|\lambda| < 1$.

ⓒ $\lambda_1 = 1/4$

$$\begin{pmatrix}3/4 & -3/16\\1 & -1/4\end{pmatrix} \sim \begin{pmatrix}12 & -3\\4 & -1\end{pmatrix} \sim \begin{pmatrix}0 & 0\\4 & -1\end{pmatrix} \Rightarrow v_1 = c\begin{pmatrix}1\\4\end{pmatrix}$$

$\lambda_2 = 3/4$

$$\begin{pmatrix}1/4 & -3/16\\1 & -3/4\end{pmatrix} \sim \begin{pmatrix}4 & -3\\4 & -3\end{pmatrix} \sim \begin{pmatrix}4 & -3\\0 & 0\end{pmatrix} \Rightarrow v_2 = d\begin{pmatrix}3\\4\end{pmatrix}$$

$$X = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = d_1 v_1 + d_2 v_2 = d_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + d_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\begin{cases} d_1 + 3d_2 = 1 \\ 4d_1 + 4d_2 = 0 \Rightarrow d_1 = -d_2 \end{cases} \Rightarrow 2d_2 = 1 \Rightarrow d_2 = 1/2$$
$$\Rightarrow d_1 = -1/2$$

$$\therefore X = \frac{-1}{2}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$A^k X = \frac{-1}{2}\left(\frac{1}{4}\right)^k \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \frac{1}{2}\left(\frac{3}{4}\right)^k \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$
$$= \frac{-1}{2^{2k+1}}\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \frac{3^k}{2^{2k+1}}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{-1}{2^{2k+1}}\left[\begin{pmatrix} 1 \\ 4 \end{pmatrix} - 3^k \begin{pmatrix} 3 \\ 4 \end{pmatrix}\right]$$

The sequence begins: $0, 1, 1, \frac{13}{16}, \frac{5}{8}, \cdots$

$$\begin{array}{r} 27 \\ \times\ 4 \\ \hline 108 \end{array}$$

$$A^1 X = \frac{-1}{8}\left[\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 9 \\ 12 \end{pmatrix}\right] = \frac{-1}{8}\left[\begin{pmatrix} -8 \\ -8 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} \checkmark$$

$\frac{80}{128} = \frac{40}{64} = \frac{10}{16} = \frac{5}{8}$

$$A^2 X = \frac{-1}{32}\left[\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 27 \\ 36 \end{pmatrix}\right] = \frac{-1}{32}\left[\begin{pmatrix} -26 \\ -32 \end{pmatrix}\right] = \begin{pmatrix} 13/16 \\ 1 \end{pmatrix} = \begin{pmatrix} a_3 \\ a_2 \end{pmatrix} \checkmark$$

$\frac{104}{128} = \frac{52}{64} = \frac{26}{32} = \frac{13}{16}$

$$A^3 X = \frac{-1}{128}\left[\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 81 \\ 108 \end{pmatrix}\right] = \frac{-1}{128}\left[\begin{pmatrix} -80 \\ -104 \end{pmatrix}\right] = \begin{pmatrix} 5/8 \\ 13/16 \end{pmatrix} = \begin{pmatrix} a_4 \\ a_3 \end{pmatrix} \checkmark$$

(d) As k gets large $\begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix} \approx \frac{3^k}{2^{2k+1}}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

so $a_k \approx \frac{3^k}{2^{2k-1}}$. Also $a_k \to 0$ as $k \to \infty$.

(4) $a_0=0,\ a_1=1,\ a_k=\left(\frac{1}{2}\right)a_{k-1}+\left(\frac{3}{16}\right)a_{k-2},\ k\geq 2$

(a) $\begin{pmatrix} a_k \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{16} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{k-1} \\ a_{k-2} \end{pmatrix}$

So $A = \begin{pmatrix} 1/2 & 3/16 \\ 1 & 0 \end{pmatrix}$.

(b) $P_A(\lambda) = \left(\frac{1}{2}-\lambda\right)(-\lambda) - \frac{3}{16} = \lambda^2 - \frac{1}{2}\lambda - \frac{3}{16} = 0$

$\lambda = \dfrac{\frac{1}{2} \pm \sqrt{\frac{1}{4}+\frac{3}{4}}}{2} = \dfrac{\frac{1}{2}\pm 1}{2} = \dfrac{-1}{4}, \dfrac{3}{4}$

So the process is stable since $|\lambda|<1$.

(c) $\underline{\lambda_1 = \frac{-1}{4}}$

$\begin{pmatrix} 3/4 & 3/16 \\ 1 & 1/4 \end{pmatrix} \sim \begin{pmatrix} 12 & 3 \\ 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix} \Rightarrow v_1 = c\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$\underline{\lambda_2 = \frac{3}{4}}$

$\begin{pmatrix} -1/4 & 3/16 \\ 1 & -3/4 \end{pmatrix} \sim \begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix} \sim \begin{pmatrix} -4 & 3 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = d\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$x = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = d_1\begin{pmatrix} -1 \\ 4 \end{pmatrix} + d_2\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\Rightarrow \begin{cases} -d_1 + 3d_2 = 1 \\ 4d_1 + 4d_2 = 0 \end{cases} \Rightarrow d_1 = -d_2 \Rightarrow 4d_2 = 1 \Rightarrow d_2 = \frac{1}{4} \Rightarrow d_1 = \frac{-1}{4}$

So: $A^k x = \dfrac{-1}{4}\left(\dfrac{-1}{4}\right)^k \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \dfrac{1}{4}\left(\dfrac{3}{4}\right)^k \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$= \dfrac{(-1)^{k+1}}{4^{k+1}}\begin{pmatrix} -1 \\ 4 \end{pmatrix} + \dfrac{3^k}{4^{k+1}}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

The sequence begins: $0, 1, \frac{1}{2}, \frac{7}{16}, \frac{5}{16}$

$$A^1x = \frac{1}{16}\begin{pmatrix}-1\\4\end{pmatrix} + \frac{3}{16}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}-1/16\\4/16\end{pmatrix} + \begin{pmatrix}9/16\\12/16\end{pmatrix} = \begin{pmatrix}8/16\\16/16\end{pmatrix} = \begin{pmatrix}1/2\\1\end{pmatrix}$$

$$= \begin{pmatrix}a_2\\a_1\end{pmatrix} \checkmark$$

$$\overset{2}{27} \times 4 = 108$$

$$A^2x = \frac{-1}{64}\begin{pmatrix}-1\\4\end{pmatrix} + \frac{9}{64}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}1/64\\-4/64\end{pmatrix} + \begin{pmatrix}27/64\\36/64\end{pmatrix} = \begin{pmatrix}28/64\\32/64\end{pmatrix} = \begin{pmatrix}7/16\\1/2\end{pmatrix}$$

$$= \begin{pmatrix}a_3\\a_2\end{pmatrix} \checkmark$$

$$\frac{80}{256} = \frac{40}{128} = \frac{20}{64} = \frac{10}{32} = 5/16$$

$$\frac{112}{256} = \frac{56}{128} = \frac{28}{64} = \frac{14}{32} = 7/16$$

$$A^3x = \frac{1}{256}\begin{pmatrix}-1\\4\end{pmatrix} + \frac{27}{256}\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}-1/256\\4/256\end{pmatrix} + \begin{pmatrix}81/256\\108/256\end{pmatrix} = \begin{pmatrix}80/256\\112/256\end{pmatrix}$$

$$= \begin{pmatrix}5/16\\7/16\end{pmatrix} = \begin{pmatrix}a_4\\a_3\end{pmatrix} \checkmark$$

(d) For large k, $\begin{pmatrix}a_{k+1}\\a_k\end{pmatrix} \approx \frac{3^k}{4^{k+1}}\begin{pmatrix}3\\4\end{pmatrix}$

so $a_k \approx \frac{3^k}{4^k}$. Also $a_k \to 0$ as $k \to \infty$.

⑧ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$P_A(\lambda) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$

$\Rightarrow \lambda = -1, 3$

$\underline{\lambda_1 = -1}$

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underline{\lambda_2 = 3}$

$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = d \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Let $C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, then $D = C^{-1}AC = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \begin{cases} y_1' = -y_1 \\ y_2' = 3y_2 \end{cases} \Rightarrow \begin{cases} y_1 = c_1 e^{-t} \\ y_2 = c_2 e^{3t} \end{cases}$

Now:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{3t} \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{3t} \\ -c_1 e^{-t} + c_2 e^{3t} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$$6(7-16)-3(14-32)-3(-16+16)$$
$$6(-9)-3(-18)=-54+54=0$$

(10) $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Ax$$

$$\begin{aligned} p_A(\lambda) &= (6-\lambda)[(-1-\lambda)(-7-\lambda)-16]-3[(-2)(-7-\lambda)-(16)(2)] \\ &\quad -3[(-2)(8)-(16)(-1-\lambda)] \\ &= (6-\lambda)[7+8\lambda+\lambda^2-16]-3[14+2\lambda-32]-3[-16+16+16\lambda] \\ &= (6-\lambda)[\lambda^2+8\lambda-9]-3[2\lambda-18]-3[16\lambda] \\ &= -3[-2\lambda^2-16\lambda+18]-\lambda^3-8\lambda^2+9\lambda-3[18\lambda-18] \\ &= -3[-2\lambda^2+2\lambda]-\lambda^3-8\lambda^2+9\lambda \\ &= -\lambda^3-2\lambda^2+3\lambda = -\lambda(\lambda^2+2\lambda-3) \\ &= -\lambda(\lambda+3)(\lambda-1)=0 \end{aligned}$$

$$\Rightarrow \lambda = -3, 0, 1$$

$\underline{\lambda_1 = -3}$

$$\begin{pmatrix} 9 & 3 & -3 \\ -2 & 2 & 2 \\ 16 & 8 & -4 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & -1 \\ -1 & 1 & 1 \\ 4 & 2 & -1 \end{pmatrix} \underset{R_3+4R_2}{\overset{R_1+2R_2}{\sim}} \begin{pmatrix} 1 & 3 & 1 \\ -1 & 1 & 1 \\ 0 & 6 & 3 \end{pmatrix}$$

$$\underset{R_3/3}{\overset{R_2+R_1}{\sim}} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\Rightarrow v_1 = c\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

$$6(7-16)-3(14-32)-3\cancel{(-16+16)}$$
$$3(2(-9))-3(-18)=0$$

$\underline{\lambda_2=0}$

$$\begin{pmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{pmatrix} \overset{R_2+R_1}{\sim} \begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 16 & 8 & -7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 16 & 8 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$\underline{\lambda_3=1}$

$$\begin{pmatrix} 5 & 3 & -3 \\ -2 & -2 & 2 \\ 16 & 8 & -8 \end{pmatrix} \sim \begin{pmatrix} 5 & 3 & -3 \\ 1 & 1 & -1 \\ 2 & 1 & -1 \end{pmatrix} \underset{R_3-2R_2}{\overset{R_1-4R_2}{\sim}} \begin{pmatrix} 4 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\underset{R_2+R_3}{\overset{R_1-R_3}{\sim}} \begin{pmatrix} 4 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \Rightarrow V_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Let

$$C = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ -2 & 0 & -1 \end{pmatrix} \quad \text{then} \quad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$y' = Dy \Rightarrow \begin{cases} y_1' = -3y_1 \\ y_2' = 0 \\ y_3' = y_3 \end{cases} \Rightarrow \begin{cases} y_1 = c_1 e^{-3t} \\ y_2 = c_2 \\ y_3 = c_3 e^{t} \end{cases}$$

So

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^{-3t} \\ c_2 \\ c_3 e^{t} \end{pmatrix} = \begin{pmatrix} -c_1 e^{-3t} + c_2 \\ c_1 e^{-3t} - 2c_2 + c_3 e^{t} \\ -2c_1 e^{-3t} - c_3 e^{t} \end{pmatrix}$$

6.1) 2) proj of (3,4) on sp((2,1)) in $\mathbb{R}^2$

24.6

$$p = \frac{(3,4)\cdot(2,1)}{(2,1)\cdot(2,1)}(2,1) = \frac{6+4}{4+1}(2,1) = (4,2)$$

4) proj of (1,2,1) on the line $x=3t, y=t,$
$z=2t$ in $\mathbb{R}^3$.

line = sp((3,1,2))

$$p = \frac{(1,2,1)\cdot(3,1,2)}{(3,1,2)\cdot(3,1,2)}(3,1,2) = \frac{3+2+2}{9+1+4}(3,1,2) = \left(\frac{3}{2}, \frac{1}{2}, 1\right)$$

6) proj of $(2,-1,3,-5)$ on $sp((1,0,-1,2))$ in $\mathbb{R}^4$

$$p = \frac{(2,-1,3,-5)\cdot(1,0,-1,2)}{(1,0,-1,2)\cdot(1,0,-1,2)}(1,0,-1,2) = \frac{2-3-10}{1+1+4}(1,0,-1,2)$$

$$= \frac{-11}{6}(1,0,-1,2) = \left(\frac{-11}{6}, 0, \frac{11}{6}, \frac{-11}{3}\right)$$