HW #6 : 7.2 # 2, 6, 11, 20, 23 b i

7.2) 2) $T(x,y) = \begin{pmatrix} 2x+3y \\ x+2y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$B = \{ \underset{b_1}{(1,-1)}, \underset{b_2}{(1,1)} \}$, $B' = \{ \underset{b_1'}{(2,3)}, \underset{b_2'}{(1,2)} \}$

$T(b_1) = (-1,-1) \quad T(b_2) = (5,3)$

Then:

$$\left(M_B \mid M_{T(B)}\right) = \left(\begin{array}{cc|cc} 1 & 1 & -1 & 5 \\ -1 & 1 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & -1 & 5 \\ 0 & 2 & -2 & 8 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 1 & -1 & 5 \\ 0 & 1 & -1 & 4 \end{array}\right) \sim \left(I \,\middle|\, \begin{array}{cc} 0 & 1 \\ -1 & 4 \end{array}\right)$$

$$\Rightarrow R_B = \begin{pmatrix} 0 & 1 \\ -1 & 4 \end{pmatrix}$$

Now $T(b_1') = (13, 8) \quad T(b_2') = (8,5)$

Then:

$$\left(M_{B'} \mid M_{T(B')}\right) = \left(\begin{array}{cc|cc} 2 & 1 & 13 & 8 \\ 3 & 2 & 8 & 5 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & 1 & 13 & 8 \\ 1 & 1 & -5 & -3 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 0 & -1 & 23 & 14 \\ 1 & 1 & -5 & -3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 0 & -1 & 23 & 14 \\ 1 & 0 & 18 & 11 \end{array}\right) \sim \left(I \,\middle|\, \begin{array}{cc} 18 & 11 \\ -23 & -14 \end{array}\right)$$

$$\Rightarrow R_{B'} = \begin{pmatrix} 18 & 11 \\ -23 & -14 \end{pmatrix}.$$ For $C = C_{B',B}$

$$\left(M_B \mid M_{B'}\right) = \left(\begin{array}{cc|cc} 1 & 1 & 2 & 1 \\ -1 & 1 & 3 & 2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 2 & 1 \\ 0 & 2 & 5 & 3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 2 & 1 \\ 0 & 1 & 5/2 & 3/2 \end{array}\right)$$

$$\sim \left(I \,\middle|\, \begin{array}{cc} -1/2 & -1/2 \\ 5/2 & 3/2 \end{array}\right) \Rightarrow C = \begin{pmatrix} -1/2 & -1/2 \\ 5/2 & 3/2 \end{pmatrix}$$

(6) T: reflection across $5x = 3y$

$B = \{(3,5),(5,-3)\}$, $B' = \{e_1, e_2\} = E$

$T(3,5) = (3,5)$, $T(5,-3) = (-5,3)$

Thus:

$$\left(\begin{array}{cc|cc} 3 & 5 & 3 & -5 \\ 5 & -3 & 5 & 3 \end{array}\right) \sim \left(\begin{array}{cc|cc} 3 & 5 & 3 & -5 \\ 2 & -8 & 2 & 8 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 3 & 5 & 3 & -5 \\ 1 & -4 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{cc|cc} 0 & 17 & 0 & -17 \\ 1 & -4 & 1 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 0 & 1 & 0 & -1 \\ 1 & -4 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{cc|cc} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\Rightarrow R_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$C = C_{B'B} = C_{BB'}^{-1} = \begin{pmatrix} 3 & 5 \\ 5 & -3 \end{pmatrix}^{-1} = \frac{-1}{34}\begin{pmatrix} -3 & -5 \\ -5 & 3 \end{pmatrix}$$

$$= \frac{1}{34}\begin{pmatrix} 3 & 5 \\ 5 & -3 \end{pmatrix}$$

Then,

$$R_{B'} = C^{-1} R_B C = \frac{1}{34}\begin{pmatrix} 3 & 5 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 5 & -3 \end{pmatrix}$$

$$= \frac{1}{34}\begin{pmatrix} 3 & 5 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix} = \frac{1}{34}\begin{pmatrix} -16 & 30 \\ 30 & 16 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} -8 & 15 \\ 15 & 8 \end{pmatrix}$$

(11) $T(p(x)) = p(x+1) + p(x)$

$B = \{x^2, x, 1\}$, $B' = \{1, x, x^2\}$

$T(x^2) = (x+1)^2 + x^2 = 2x^2 + 2x + 1$

$T(x) = (x+1) + x = 2x + 1$

$T(1) = 1 + 1 = 2$

Then

$$R_B = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \Rightarrow R_{B'} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow C = C_{B',B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

---

(20) $T(x_1, x_2, x_3) = (x_1, 4x_2 + 7x_3, 2x_2 - x_3)$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 7 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A\vec{x}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 4-\lambda & 7 \\ 0 & 2 & -1-\lambda \end{vmatrix} = (1-\lambda)\left[(4-\lambda)(-1-\lambda) - 14\right]$$

$$= (1-\lambda)(-4 + \lambda - 4\lambda + \lambda^2 - 14) = (1-\lambda)(-18 - 3\lambda + \lambda^2)$$

$$= (1-\lambda)(-6+\lambda)(3+\lambda) = 0$$

$\Rightarrow \lambda = -3, 1, 6$

$\underline{\lambda_1 = -3:}$

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 7 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_{-3} = sp\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$$

$\underline{\lambda_2 = 1}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 7 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 10 \\ 0 & 1 & -1 \end{pmatrix}$

$\Rightarrow E_1 = sp\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

$\underline{\lambda_3 = 6}$ $\begin{pmatrix} -5 & 0 & 0 \\ 0 & -2 & 7 \\ 0 & 2 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -7 \end{pmatrix}$

$\Rightarrow E_6 = sp\left\{\begin{pmatrix} 0 \\ 7 \\ 2 \end{pmatrix}\right\}$

T is diagonalizable.

---

23 b True. Representations are unique.

i False. Consider $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Similar matrices have the same eigenvalues.