HW # 7

8.1 # 2, 6, 10, 12

8.2 # 2, 6, 9, 10abc

8.1) ② $8x^2+9xy-3y^2 = 8x^2+\frac{9}{2}(xy+yx)-3y^2$

So $U=\begin{pmatrix}8&9\\0&-3\end{pmatrix}$ & $A=\begin{pmatrix}8&9/2\\9/2&-3\end{pmatrix}$

⑥ $(x,y)\begin{pmatrix}7&-10\\15&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 7x^2+5xy+2y^2$

$= 7x^2+\frac{5}{2}(xy+yx)+2y^2$

So $U=\begin{pmatrix}7&5\\0&2\end{pmatrix}$ & $A=\begin{pmatrix}7&5/2\\5/2&2\end{pmatrix}$

⑩ $3x^2+4xy \Rightarrow U=\begin{pmatrix}3&4\\0&0\end{pmatrix} \Rightarrow A=\begin{pmatrix}3&2\\2&0\end{pmatrix}$

$|A-\lambda I| = \begin{vmatrix}3-\lambda&2\\2&-\lambda\end{vmatrix} = -3\lambda+\lambda^2-4 = (\lambda-4)(\lambda+1)=0$

$\Rightarrow \lambda = -1, 4$

$\underline{\lambda_1=-1}$

$\begin{pmatrix}4&2\\2&1\end{pmatrix}\sim\begin{pmatrix}2&1\\2&1\end{pmatrix}\sim\begin{pmatrix}2&1\\0&0\end{pmatrix} \Rightarrow v_1=\begin{pmatrix}-1\\2\end{pmatrix}$

$\Rightarrow n_1 = \frac{1}{\sqrt5}\begin{pmatrix}-1\\2\end{pmatrix}$

$\underline{\lambda_2=4}$

$\begin{pmatrix}-1&2\\2&-4\end{pmatrix}\sim\begin{pmatrix}1&-2\\1&-2\end{pmatrix}\sim\begin{pmatrix}1&-2\\0&0\end{pmatrix} \Rightarrow v_2=\begin{pmatrix}2\\1\end{pmatrix}$

$\Rightarrow n_2 = \frac{1}{\sqrt5}\begin{pmatrix}2\\1\end{pmatrix}$

Then use

$$\begin{pmatrix} x \\ y \end{pmatrix} = C \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{5}} u + \frac{2}{\sqrt{5}} v \\ \frac{2}{\sqrt{5}} u + \frac{1}{\sqrt{5}} v \end{pmatrix}$$

to diagonalize, so:

$$(u \; v) \, C^T A C \begin{pmatrix} u \\ v \end{pmatrix} = (u \; v) \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$= -u^2 + 4v^2$$

8.2) ② $2xy - 2\sqrt{2}y = 1$

First look @ quadratic part: $2xy = 1$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. e.vals $\lambda = \pm 1$

$\underline{\lambda_1 = -1}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow n_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\underline{\lambda_2 = 1}$

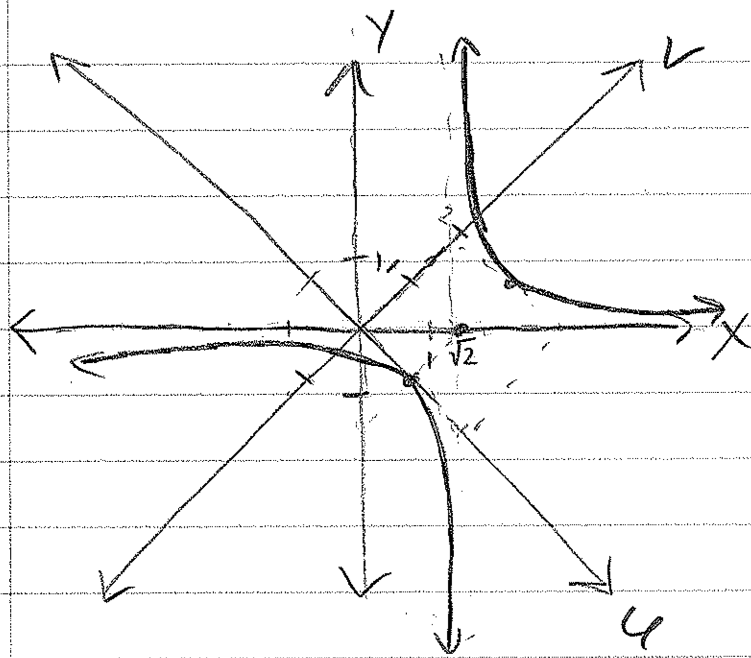
$$\Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow n_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Let $\begin{pmatrix} u \\ v \end{pmatrix} = C \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2}\, u + 1/\sqrt{2}\, v \\ -1/\sqrt{2}\, u + 1/\sqrt{2}\, v \end{pmatrix}$

Then

$$2xy - 2\sqrt{2}y = -u^2 + v^2 + 2u - 2v = -(u-1)^2 + (v-1)^2 = 1$$



⑥ $5x^2+4xy+2y^2=-1$

$$A=\begin{pmatrix}5&2\\2&2\end{pmatrix}$$

$$\begin{vmatrix}5-\lambda&2\\2&2-\lambda\end{vmatrix}=10-7\lambda+\lambda^2-4=\lambda^2-7\lambda+6=(\lambda-1)(\lambda-6)=0$$

$\lambda=1,6$

$\underline{\lambda_1=1}$

$$\begin{pmatrix}4&2\\2&1\end{pmatrix}\sim\begin{pmatrix}2&1\\0&0\end{pmatrix}\Rightarrow v_1=\begin{pmatrix}-1\\2\end{pmatrix}\Rightarrow n_1=\frac{1}{\sqrt5}\begin{pmatrix}-1\\2\end{pmatrix}$$

$\underline{\lambda_2=6}$

$$\begin{pmatrix}-1&2\\2&-4\end{pmatrix}\sim\begin{pmatrix}1&-2\\0&0\end{pmatrix}\Rightarrow v_2=\begin{pmatrix}2\\1\end{pmatrix}\Rightarrow n_2=\frac{1}{\sqrt5}\begin{pmatrix}2\\1\end{pmatrix}$$

So let $\begin{pmatrix}u\\v\end{pmatrix}=C\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{\sqrt5}\begin{pmatrix}-1&2\\2&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{-1}{\sqrt5}x+\frac{2}{\sqrt5}y\\\frac{2}{\sqrt5}x+\frac{1}{\sqrt5}y\end{pmatrix}$

Then $5x^2+4xy+2y^2=u^2+6v^2=-1$
which is an empty ellipse (in $\mathbb{R}^2$).

9 $ax^2+bxy+cy^2+dx+ey+f=0$

$A=\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$

$|A-\lambda I| = \lambda^2-(a+c)\lambda+\left(ac-\frac{b^2}{4}\right)=0$

$\Rightarrow \lambda = \dfrac{a+c \pm \sqrt{(a+c)^2-4(ac-\frac{1}{4}b^2)}}{2}$

$= \dfrac{a+c \pm \sqrt{(a+c)^2+(b^2-4ac)}}{2}$

λ is real since A is symmetric.

① If $b^2-4ac<0$ then both eigenvalues have the same sign $\Rightarrow \lambda_1\lambda_2>0$.
Thm 8.2 $\Rightarrow$ poss. deg. ellipse.

② If $b^2-4ac>0$ then the e.vals have opp. signs $\Rightarrow \lambda_1\lambda_2<0$
$\Rightarrow$ poss. deg. parabola

③ If $b^2-4ac=0$ the one eval is zero
$\Rightarrow \lambda_1\lambda_2=0 \Rightarrow$ poss. deg parabola

10 (a) $2x^2+8xy+8y^2-3x+2y=13$
$b^2-4ac=64-4\cdot2\cdot8=0$
parabola

(b) $y^2+4xy-x^2-3x=12$
$16-4\cdot(-1)\cdot1=20>0$
hyperbola

(c) $-x^2+5xy-7y^2-4y+11=0$
$b^2-4ac=25-4(-1)(-7)$
$=25-28<0$
ellipse.