HW 9

9.2 # 6, 20b, 21c, 24, 28, 33eh, 34

9.3 # 2, 6, 14, 19ef, 23

9.2) (6) $A=\begin{pmatrix} i & 1+i \\ 1 & i \end{pmatrix}$

$$A^{-1}=\frac{1}{\det A}\begin{pmatrix} i & -1 \\ -1-i & i \end{pmatrix}=\frac{1}{i^2-1-i}\begin{pmatrix} i & -1 \\ -1-i & i \end{pmatrix}=\frac{1}{-2-i}\begin{pmatrix} i & -1 \\ -1-i & i \end{pmatrix}$$

$$=\frac{1}{5}\begin{pmatrix} -1-2i & 3+i \\ 2-i & -1-2i \end{pmatrix}$$

$\frac{-1}{5}(-2+i)(1-i)$

$\frac{-1}{5}(-2+i+3i-i^2)$

$\frac{-1}{5}(-1+4i)$

$\frac{-5}{5}i+\frac{1}{5}-\frac{4}{5}i$

$=\frac{1}{5}(1-9i)$

$\frac{-1}{5}(1+2i)(1-i)$

$\frac{-1}{5}(1+2i-i-2i^2)$

$\frac{-1}{5}(3+i)$

(20b) $(1+i, 1-i, 1+i)=v$

$\bar{v}=(1-i, 1+i, 1-i)$

$\bar{v}\cdot v = 1+1+1+1+1+1=6$

$\therefore \|v\|=\sqrt{6}$

(21c) $(\underbrace{1+i, 2-i}_{=v}), (\underbrace{3i, 3+i}_{=w})$

$\langle v,w\rangle = (1-i)(3i)+(2+i)(3+i)=3i+3+6+5i-1$

$= 8+8i$ not $\perp$

$$\frac{3i}{1+i}\cdot\frac{1-i}{1-i}=\frac{3i+3}{2}=\frac{3}{2}+\frac{3}{2}i$$

$$\left(\frac{3}{2}+\frac{3}{2}i\right)(2-i)=3+3i-\frac{3}{2}i+\frac{3}{2}=\frac{9}{2}+\frac{3}{2}i$$

not $\parallel$

(24) $v=(2-i, 1+i)$. Let $u=(1-i, -2-i)$.

Then

$$\langle v,u\rangle = (2+i)(1-i) + (1-i)(-2-i)$$
$$= 2 - \cancel{i} + 1 - 2 + \cancel{2i} - \cancel{i} - 1$$
$$= 0$$

Now normalize u:

$$\|u\| = (1+1+4+1)^{1/2} = \sqrt{7}$$

$$u = \frac{1}{\sqrt{7}}(1-i, -2-i).$$

$(1-i)(-1-i)$

$-1-i+i+i^2$

-2

(28) Take $b_1 = (1-i, 1+i, 1+i)$. Let $v_2 = (1, 1, -1-i)$

$$b_2 = v_2 - \frac{\langle b_1, v_2\rangle}{\langle b_1, b_1\rangle} b_1 = (1,1,-1-i) - \frac{1+i+1-i-2}{6}(1-i, 1+i, 1+i)$$

$$= (1,1,-1-i) - 0$$

$(1+i)(1+i)$

$1-1+\boxed{2i}$

$(1+i)(-1-i)$

$-1-i-i+1$

$-2i$

So $b_2 = (1, 1, -1-i)$. Let $v_3 = (1, i, -i)$

$$b_3 = v_3 - \frac{\langle b_1, v_3\rangle}{\langle b_1, b_1\rangle} b_1 - \frac{\langle b_2, v_3\rangle}{\langle b_2, b_2\rangle} b_2$$

$$= (1, i, -i) - \frac{1+i+\cancel{1+i}\cancel{-1-i}}{6}(1-i, 1+i, 1+i) - \frac{1+i+i+1}{4}(1, 1, -1-i)$$

$$= (1, i, -i) - \frac{1+i}{6}(1-i, 1+i, 1+i) - \frac{1+i}{2}(1, 1, -1-i)$$

$$= \left(1 - \tfrac{1}{3} - \tfrac{1}{2} - \tfrac{i}{2},\ i - \tfrac{1}{3}i - \tfrac{1}{2} - \tfrac{i}{2},\ -i - \tfrac{i}{3} + i\right)$$

$$= \left(\tfrac{1}{6} - \tfrac{1}{2}i,\ -\tfrac{1}{2} + \tfrac{1}{6}i,\ -\tfrac{1}{3}i\right).$$ Mult by 6:

Orthog. basis:

$$\{(1-i, 1+i, 1+i), (1, 1, -1-i), (1-3i, -3+i, -2i)\}$$

(33) (e) $U^*U = \bar{U}^T U = I \qquad \bar{U}^T = U^{-1}$

$\bar{U} = (U^{-1})^T = (U^T)^{-1} \Rightarrow U^T = (\bar{U})^{-1}$ ✓

True

(h) $\boxed{\text{True}}$

(34) Let $\{v_1, \cdots, v_n\}$ be a subset of $\mathbb{C}^n$.

Then $\sum_{i=1}^{n} a_i v_i = 0 \iff \sum_{i=1}^{n} \bar{a}_i \bar{v}_i = 0$

since $\overline{a+b} = \bar{a} + \bar{b}$. Let $\mathcal{B}_2 = \{\bar{v}_1, \cdots, \bar{v}_n\}$

So every dependence relation in $\mathcal{B}_1$ gives a dependence relation in $\mathcal{B}_2$ and vice-versa.

$\therefore$ $\mathcal{B}_1$ is independent iff $\mathcal{B}_2$ is.

$\therefore$ Since n independent vectors in $\mathbb{C}^n$ form a basis:

$\mathcal{B}_1$ is a basis $\iff$ $\mathcal{B}_2$ is a basis.

9.3) (2) $A = \begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix}$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2i \\ -2i & 1-\lambda \end{vmatrix} = 1 - 2\lambda + \lambda^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$

$\lambda = -1, 3$

$\underline{\lambda_1 = -1}$

$\begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix} \sim \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} i \\ -1 \end{pmatrix}$

$\underline{\lambda_2 = 3}$

$$\begin{pmatrix} -2 & 2i \\ -2i & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -i \\ 1 & -i \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$\|v_1\| = \sqrt{2} = \|v_2\|$

So let $U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$ then $D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$

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$$A = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$|A - \lambda I| = (1-\lambda)(1-\lambda)(2-\lambda) + i(i)(2-\lambda)$$
$$= (2-\lambda)(\cancel{1} - 2\lambda + \lambda^2 - \cancel{1}) = (2-\lambda)(\lambda^2 - 2\lambda)$$
$$= \lambda(2-\lambda)(\lambda-2)$$

$\Rightarrow \lambda_1 = 0,\ \lambda_2, \lambda_3 = 2$

$\underline{\lambda_1 = 0}$

$$\begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -i & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$

$\underline{\lambda_2 = \lambda_3 = 2}$

$$\begin{pmatrix} -1 & -i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & -i & 0 \\ -1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} i \\ -1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\|v_1\| = \|v_2\| = \sqrt{2} \quad \|v_3\| = 1$

Let
$$U=\frac{1}{\sqrt{2}}\begin{pmatrix} i & i & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \quad \text{then} \quad D=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(14) $A=\begin{pmatrix} i & a \\ b & i \end{pmatrix}$, $a, b \in \mathbb{C}$.

$$A^*=\begin{pmatrix} -i & \bar{b} \\ \bar{a} & -i \end{pmatrix}$$

A unitarily diagonalizable $\Leftrightarrow A^*A = AA^*$

$$A^*A=\begin{pmatrix} -i & \bar{b} \\ \bar{a} & -i \end{pmatrix}\begin{pmatrix} i & a \\ b & i \end{pmatrix}=\begin{pmatrix} 1+\bar{b}b & -ia+i\bar{b} \\ i\bar{a}-ib & \bar{a}a+1 \end{pmatrix}$$

$$AA^*=\begin{pmatrix} i & a \\ b & i \end{pmatrix}\begin{pmatrix} -i & \bar{b} \\ \bar{a} & -i \end{pmatrix}=\begin{pmatrix} 1+a\bar{a} & i\bar{b}-ia \\ -ib+i\bar{a} & b\bar{b}+1 \end{pmatrix}$$

So $A^*A = AA^* \Leftrightarrow a\bar{a} = b\bar{b}$.

(19) (e) True. Since $[a_{ij}] = [a_{ji}]$ & $a_{ij} \in \mathbb{R}$.

(f) False.

$$A=\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad A^*=\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$AA^*=\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix} \quad A^*A=\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}$$

(23) ($\Longrightarrow$) Suppose A is unit. diag. $\Rightarrow A^*A = AA^*$.
Then $\forall v \in \mathbb{C}^n$ $\|Av\|^2 = (Av)^*Av = v^*A^*Av = v^*AA^*v$
$= (A^*v)^*A^*v = \|A^*v\|^2$. $\therefore \|Av\| = \|A^*v\|$

($\Leftarrow$) Suppose $\|Av\| = \|A^*v\|$. Then $\|Av\|^2 = \|A^*v\|^2$.

Thus $v^*A^*Av = v^*AA^*v$

$\Rightarrow v^*(A^*A - AA^*)v = 0 \quad \forall v \in \mathbb{C}^n$

Let $C = A^*A - AA^*$ and suppose $v^*Cv = 0 \ \forall v \in \mathbb{C}^n$.

Claim: $C = 0$.

pf| Let λ be any eigenvalue of C. Then $Cw = \lambda w$ for some $w \in \mathbb{C}^n$, $w \neq 0$.

Then $w^*Cw = w^*\lambda w = \lambda w^*w = 0 \Rightarrow \lambda = 0$ since $w \neq 0$. Thus the only eigenvalue of C is 0. Notice that:

$$C^* = (A^*A - AA^*)^* = (A^*A)^* - (AA^*)^* = A^*A^{**} - A^{**}A^*$$
$$= A^*A - AA^* = C.$$

Thus C is unitarily equivalent to a diagonal matrix all of whose diagonal entries are e.vals of C and thus zero. Thus

$$0 = U^*CU$$

for some U, unitary. Now

$U^*CU = 0 \Rightarrow C = U0U^* = 0 \Rightarrow A^*A - AA^* = 0$

$\Rightarrow A^*A = AA^*$

$\therefore A$ is normal

$\therefore A$ is unit. diag.

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