Theorem 1 (Schröder-Bernstein Theorem)**.** *Suppose that $f : A \to B$ and $g : B \to A$ are $one-to-one$ maps. Then there is a bijection between A and B.*

Proof. Define the sets A_n and B_n as follows:

$$A_0 = A, \qquad B_0 = B,$$
$$A_{n+1} = g \circ f(A_n), \quad B_{n+1} = f \circ g(B_n).$$

By induction on n we have that:

$$A_n \supset g(B_n) \supset A_{n+1},$$

and

$$B_n \supset f(A_n) \supset B_{n+1},$$

thus giving us the chain of incusions:

$$A_0 \supset g(B_0) \supset A_1 \supset g(B_1) \supset A_2 \supset \cdots,$$

and

$$B_0 \supset f(A_0) \supset B_1 \supset f(A_1) \supset B_2 \supset \cdots.$$

Define the sets A_∞ and B_∞ by:

$$A_\infty = \bigcap_{n=0}^{\infty} A_n \; and \; B_\infty = \bigcap_{n=0}^{\infty} B_n.$$

This gives that

$$B_\infty = \bigcap_{n=0}^{\infty} B_n \supset \bigcap_{n=0}^{\infty} f(A_n) \supset \bigcap_{n=0}^{\infty} B_{n+1} = B_\infty.$$

Using the fact that f is $1-1$ we get:

$$f(A_\infty) = f(\bigcap_{n=0}^{\infty} A_n) = \bigcap_{n=0}^{\infty} f(A_n) = \bigcap_{n=0}^{\infty} B_n = B_\infty.$$

Thus we have that f maps A_∞ *onto* B_∞, which means that f is a bijection between A_∞ and B_∞. Now we write A and B as a disjoint union as follows:

$$A = A_\infty \cup [A_0 \setminus g(B_0)] \cup [g(B_0) \setminus A_1] \cup [A_1 \setminus g(B_1)] \cup [g(B_1) \setminus A_2] \cup \cdots,$$
$$B = B_\infty \cup [B_0 \setminus f(A_0)] \cup [f(A_0) \setminus B_1] \cup [B_1 \setminus f(A_1)] \cup [f(A_1) \setminus B_2] \cup \cdots.$$

Thus all that remains to be checked is that, for all n:

$$f[A_n \setminus g(B_n)] = f(A_n) \setminus B_{n+1}$$

and

$$g[B_n \setminus f(A_n)] = g(B_n) \setminus A_{n+1}.$$

For any given n we have (since f and g are $1-1$):

$$f[A_n \setminus g(B_n)] = f(A_n) \setminus f(g(A_n)) = f(A_n) \setminus B_{n+1}$$

and

$$g[B_n \setminus f(A_n)] = g(B_n) \setminus g(f(B_n)) = g(B_n) \setminus A_{n+1}.$$

Thus we can construct the bijection $\zeta : A \to B$ by:

$$\zeta(x) = \begin{cases} f(x), & x \in A_\infty \; or \; x \in A_n \setminus g(B_n) \; for \; some \; n, \\ g^{-1}(x) & x \notin A_\infty \; and \; x \in g(B_n) \setminus A_{n+1} \; for \; some \; n. \end{cases}$$

□