Case I N contains a 3-cycle.

Case II N contains a product of disjoint cycles, at least one of which has length greater than 3. [*Hint:* Suppose N contains the disjoint product $\sigma = \mu(a_1, a_2, \cdots, a_r)$. Show $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$ is in N, and compute it.]

Case III N contains a disjoint product of the form $\sigma = \mu(a_4, a_5, a_6)(a_1, a_2, a_3)$. [*Hint:* Show $\sigma^{-1}(a_1, a_2, a_4)$ $\sigma(a_1, a_2, a_4)^{-1}$ is in N, and compute it.]

Case IV N contains a disjoint product of the form $\sigma = \mu(a_1, a_2, a_3)$ where μ is a product of disjoint 2-cycles. [*Hint:* Show $\sigma^2 \in N$ and compute it.]

Case V N contains a disjoint product σ of the form $\sigma = \mu(a_3, a_4)(a_1, a_2)$, where μ is a product of an even number of disjoint 2-cycles. [*Hint:* Show that $\sigma^{-1}(a_1, a_2, a_3)\sigma(a_1, a_2, a_3)^{-1}$ is in N, and compute it to deduce that $\alpha = (a_2, a_4)(a_1, a_3)$ is in N. Using $n \geq 5$ for the first time, find $i \neq a_1, a_2, a_3, a_4$ in $\{1, 2, \cdots, n\}$. Let $\beta = (a_1, a_3, i)$. Show that $\beta^{-1}\alpha\beta\alpha \in N$, and compute it.]

40. Let N be a normal subgroup of G and let H be any subgroup of G. Let $HN = \{hn \mid h \in H, n \in N\}$. Show that HN is a subgroup of G, and is the smallest subgroup containing both N and H.

41. With reference to the preceding exercise, let M also be a normal subgroup of G. Show that NM is again a normal subgroup of G.

42. Show that if H and K are normal subgroups of a group G such that $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H$ and $k \in K$. [*Hint:* Consider the commutator $hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} = h(kh^{-1}k^{-1})$.]

SECTION 16 †Group Action on a Set

We have seen examples of how groups may *act on things*, like the group of symmetries of a triangle or of a square, the group of rotations of a cube, the general linear group acting on $\mathbb{R}^n$, and so on. In this section, we give the general notion of group action on a set. The next section will give an application to counting.

The Notion of a Group Action

Definition 2.1 defines a binary operation $*$ on a set S to be a function mapping $S \times S$ into S. The function $*$ gives us a rule for "multiplying" an element s_1 in S and an element s_2 in S to yield an element $s_1 * s_2$ in S.

More generally, for any sets A, B, and C, we can view a map $* : A \times B \to C$ as defining a "multiplication," where any element a of A times any element b of B has as value some element c of C. Of course, we write $a * b = c$, or simply $ab = c$. In this section, we will be concerned with the case where X is a set, G is a group, and we have a map $* : G \times X \to X$. We shall write $*(g, x)$ as $g * x$ or gx.

16.1 Definition Let X be a set and G a group. An **action of G on X** is a map $* : G \times X \to X$ such that

1. $ex = x$ for all $x \in X$,
2. $(g_1g_2)(x) = g_1(g_2x)$ for all $x \in X$ and all $g_1, g_2 \in G$.

Under these conditions, X is a **G-set.** ■

† This section is a prerequisite only for Sections 17 and 36.

16.2 Example Let X be any set, and let H be a subgroup of the group S_X of all permutations of X. Then X is an H-set, where the action of $\sigma \in H$ on X is its action as an element of S_X, so that $\sigma x = \sigma(x)$ for all $x \in X$. Condition 2 is a consequence of the definition of permutation multiplication as function composition, and Condition 1 is immediate from the definition of the identity permutation as the identity function. Note that, in particular, $\{1, 2, 3, \cdots, n\}$ is an S_n set. ▲

Our next theorem will show that for every G-set X and each $g \in G$, the map $\sigma_g : X \to X$ defined by $\sigma_g(x) = gx$ is a permutation of X, and that there is a homomorphism $\phi : G \to S_X$ such that the action of G on X is essentially the Example 16.2 action of the image subgroup $H = \phi[G]$ of S_X on X. So actions of subgroups of S_X on X describe all possible group actions on X. When studying the set X, actions using subgroups of S_X suffice. However, sometimes a set X is used to study G via a group action of G on X. Thus we need the more general concept given by Definition 16.1.

16.3 Theorem Let X be a G-set. For each $g \in G$, the function $\sigma_g : X \to X$ defined by $\sigma_g(x) = gx$ for $x \in X$ is a permutation of X. Also, the map $\phi : G \to S_X$ defined by $\phi(g) = \sigma_g$ is a homomorphism with the property that $\phi(g)(x) = gx$.

Proof To show that σ_g is a permutation of X, we must show that it is a one-to-one map of X onto itself. Suppose that $\sigma_g(x_1) = \sigma_g(x_2)$ for $x_1, x_2 \in X$. Then $gx_1 = gx_2$. Consequently, $g^{-1}(gx_1) = g^{-1}(gx_2)$. Using Condition 2 in Definition 16.1, we see that $(g^{-1}g)x_1 = (g^{-1}g)x_2$, so $ex_1 = ex_2$. Condition 1 of the definition then yields $x_1 = x_2$, so σ_g is one to one. The two conditions of the definition show that for $x \in X$, we have $\sigma_g(g^{-1}x) = g(g^{-1})x = (gg^{-1})x = ex = x$, so σ_g maps X onto X. Thus σ_g is indeed a permutation.

To show that $\phi : G \to S_X$ defined by $\phi(g) = \sigma_g$ is a homomorphism, we must show that $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$ for all $g_1, g_2 \in G$. We show the equality of these two permutations in S_X by showing they both carry an $x \in X$ into the same element. Using the two conditions in Definition 16.1 and the rule for function composition, we obtain

$$\begin{aligned}\phi(g_1g_2)(x) &= \sigma_{g_1g_2}(x) = (g_1g_2)x = g_1(g_2x) = g_1\sigma_{g_2}(x) = \sigma_{g_1}(\sigma_{g_2}(x))\\ &= (\sigma_{g_1} \circ \sigma_{g_2})(x) = (\sigma_{g_1}\sigma_{g_2})(x) = (\phi(g_1)\phi(g_2))(x).\end{aligned}$$

Thus ϕ is a homomorphism. The stated property of ϕ follows at once since by our definitions, we have $\phi(g)(x) = \sigma_g(x) = gx$. ◆

It follows from the preceding theorem and Theorem 13.15 that if X is G-set, then the subset of G leaving every element of X fixed is a normal subgroup N of G, and we can regard X as a G/N-set where the action of a coset gN on X is given by $(gN)x = gx$ for each $x \in X$. If $N = \{e\}$, then the identity element of G is the only element that leaves every $x \in X$ fixed; we then say that G **acts faithfully** on X. A group G is **transitive** on a G-set X if for each $x_1, x_2 \in X$, there exists $g \in G$ such that $gx_1 = x_2$. Note that G is transitive on X if and only if the subgroup $\phi[G]$ of S_X is transitive on X, as defined in Exercise 49 of Section 8.

We continue with more examples of G-sets.

16.4 Example Every group G is itself a G-set, where the action on $g_2 \in G$ by $g_1 \in G$ is given by left multiplication. That is, $*(g_1, g_2) = g_1 g_2$. If H is a subgroup of G, we can also regard G as an H-set, where $*(h, g) = hg$. ▲

16.5 Example Let H be a subgroup of G. Then G is an H-set under conjugation where $*(h, g) = hgh^{-1}$ for $g \in G$ and $h \in H$. Condition 1 is obvious, and for Condition 2 note that

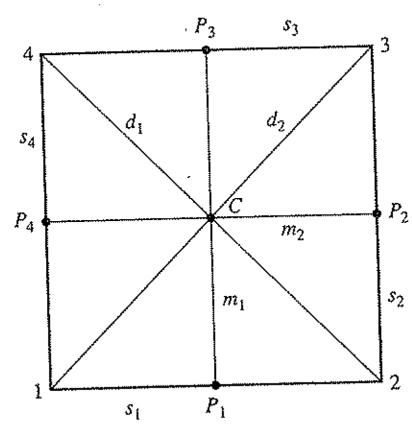
$$*(h_1h_2, g) = (h_1h_2)g(h_1h_2)^{-1} = h_1(h_2gh_2^{-1})h_1^{-1} = *(h_1, *(h_2, g)).$$

We always write this action of H on G by conjugation as hgh^{-1}. The abbreviation hg described before the definition would cause terrible confusion with the group operation of G. ▲

16.6 Example For students who have studied vector spaces with real (or complex) scalars, we mention that the axioms $(rs)\mathbf{v} = r(s\mathbf{v})$ and $1\mathbf{v} = \mathbf{v}$ for scalars r and s and a vector $\mathbf{v}$ show that the set of vectors is an $\mathbb{R}^*$-set (or a $\mathbb{C}^*$-set) for the multiplicative group of nonzero scalars. ▲

16.7 Example Let H be a subgroup of G, and let L_H be the set of all left cosets of H. Then L_H is a G-set, where the action of $g \in G$ on the left coset xH is given by $g(xH) = (gx)H$. Observe that this action is well defined: if $yH = xH$, then $y = xh$ for some $h \in H$, and $g(yH) = (gy)H = (gxh)H = (gx)(hH) = (gx)H = g(xH)$. A series of exercises shows that every G-set is isomorphic to one that may be formed using these left coset G-sets as building blocks. (See Exercises 14 through 17.) ▲

16.8 Example Let G be the group $D_4 = \{\rho_0, \rho_1, \rho_2, \rho_3, \mu_1, \mu_2, \delta_1, \delta_2\}$ of symmetries of the square, described in Example 8.10. In Fig. 16.9 we show the square with vertices 1, 2, 3, 4 as in Fig. 8.11. We also label the sides s_1, s_2, s_3, s_4, the diagonals d_1 and d_2, vertical and horizontal axes m_1 and m_2, the center point C, and midpoints P_i of the sides s_i. Recall that ρ_i corresponds to rotating the square counterclockwise through $\pi i/2$ radians, μ_i



16.9 Figure

16.10 Table

	1	2	3	4	s_1	s_2	s_3	s_4	m_1	m_2	d_1	d_2	C	P_1	P_2	P_3	P_4
ρ_0	1	2	3	4	s_1	s_2	s_3	s_4	m_1	m_2	d_1	d_2	C	P_1	P_2	P_3	P_4
ρ_1	2	3	4	1	s_2	s_3	s_4	s_1	m_2	m_1	d_2	d_1	C	P_2	P_3	P_4	P_1
ρ_2	3	4	1	2	s_3	s_4	s_1	s_2	m_1	m_2	d_1	d_2	C	P_3	P_4	P_1	P_2
ρ_3	4	1	2	3	s_4	s_1	s_2	s_3	m_2	m_1	d_2	d_1	C	P_4	P_1	P_2	P_3
μ_1	2	1	4	3	s_1	s_4	s_3	s_2	m_1	m_2	d_2	d_1	C	P_1	P_4	P_3	P_2
μ_2	4	3	2	1	s_3	s_2	s_1	s_4	m_1	m_2	d_2	d_1	C	P_3	P_2	P_1	P_4
δ_1	3	2	1	4	s_2	s_1	s_4	s_3	m_2	m_1	d_1	d_2	C	P_2	P_1	P_4	P_3
δ_2	1	4	3	2	s_4	s_3	s_2	s_1	m_2	m_1	d_1	d_2	C	P_4	P_3	P_2	P_1

corresponds to flipping on the axis m_i, and δ_i to flipping on the diagonal d_i. We let

$$X = \{1, 2, 3, 4, s_1, s_2, s_3, s_4, m_1, m_2, d_1, d_2, C, P_1, P_2, P_3, P_4\}.$$

Then X can be regarded as a D_4-set in a natural way. Table 16.10 describes completely the action of D_4 on X and is given to provide geometric illustrations of ideas to be introduced. We should be sure that we understand how this table is formed before continuing. ▲

Isotropy Subgroups

Let X be a G-set. Let $x \in X$ and $g \in G$. It will be important to know when $gx = x$. We let

$$X_g = \{x \in X \mid gx = x\} \qquad \text{and} \qquad G_x = \{g \in G \mid gx = x\}.$$

16.11 Example For the D_4-set X in Example 16.8, we have

$$X_{\rho_0} = X, \qquad X_{\rho_1} = \{C\}, \qquad X_{\mu_1} = \{s_1, s_3, m_1, m_2, C, P_1, P_3\}$$

Also, with $G = D_4$,

$$G_1 = \{\rho_0, \delta_2\}, \qquad G_{s_3} = \{\rho_0, \mu_1\}, \qquad G_{d_1} = \{\rho_0, \rho_2, \delta_1, \delta_2\}.$$

We leave the computation of the other X_σ and G_x to Exercises 1 and 2. ▲

Note that the subsets G_x given in the preceding example were, in each case, subgroups of G. This is true in general.

16.12 Theorem Let X be a G-set. Then G_x is a subgroup of G for each $x \in X$.

Proof Let $x \in X$ and let $g_1, g_2 \in G_x$. Then $g_1x = x$ and $g_2x = x$. Consequently, $(g_1g_2)x = g_1(g_2x) = g_1x = x$, so $g_1g_2 \in G_x$, and G_x is closed under the induced operation of G. Of course $ex = x$, so $e \in G_x$. If $g \in G_x$, then $gx = x$, so $x = ex = (g^{-1}g)x = g^{-1}(gx) = g^{-1}x$, and consequently $g^{-1} \in G_x$. Thus G_x is a subgroup of G. ◆

16.13 Definition Let X be a G-set and let $x \in X$. The subgroup G_x is the **isotropy subgroup of x**. ■

Orbits

For the D_4-set X of Example 16.8 with action table in Table 16.10, the elements in the subset $\{1, 2, 3, 4\}$ are carried into elements of this same subset under action by D_4. Furthermore, each of the elements 1, 2, 3, and 4 is carried into all the other elements of the subset by the various elements of D_4. We proceed to show that every G-set X can be partitioned into subsets of this type.

16.14 Theorem Let X be a G-set. For $x_1, x_2 \in X$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $gx_1 = x_2$. Then $\sim$ is an equivalence relation on X.

Proof For each $x \in X$, we have $ex = x$, so $x \sim x$ and $\sim$ is reflexive.

Suppose $x_1 \sim x_2$, so $gx_1 = x_2$ for some $g \in G$. Then $g^{-1}x_2 = g^{-1}(gx_1) = (g^{-1}g)x_1 = ex_1 = x_1$, so $x_2 \sim x_1$, and $\sim$ is symmetric.

Finally, if $x_1 \sim x_2$ and $x_2 \sim x_3$, then $g_1x_1 = x_2$ and $g_2x_2 = x_3$ for some $g_1, g_2 \in G$. Then $(g_2g_1)x_1 = g_2(g_1x_1) = g_2x_2 = x_3$, so $x_1 \sim x_3$ and $\sim$ is transitive. ◆

16.15 Definition Let X be a G-set. Each cell in the partition of the equivalence relation described in Theorem 16.14 is an **orbit in X under G**. If $x \in X$, the cell containing x is the **orbit of x**. We let this cell be Gx. ■

The relationship between the orbits in X and the group structure of G lies at the heart of the applications that appear in Section 17. The following theorem gives this relationship. Recall that for a set X, we use $|X|$ for the number of elements in X, and $(G : H)$ is the index of a subgroup H in a group G.

16.16 Theorem Let X be a G-set and let $x \in X$. Then $|Gx| = (G : G_x)$. If $|G|$ is finite, then $|Gx|$ is a divisor of $|G|$.

Proof We define a one-to-one map ψ from Gx onto the collection of left cosets of G_x in G. Let $x_1 \in Gx$. Then there exists $g_1 \in G$ such that $g_1x = x_1$. We define $\psi(x_1)$ to be the left coset g_1G_x of G_x. We must show that this map ψ is well defined, independent of the choice of $g_1 \in G$ such that $g_1x = x_1$. Suppose also that $g_1'x = x_1$. Then, $g_1x = g_1'x$, so $g_1^{-1}(g_1x) = g_1^{-1}(g_1'x)$, from which we deduce $x = (g_1^{-1}g_1')x$. Therefore $g_1^{-1}g_1' \in G_x$, so $g_1' \in g_1G_x$, and $g_1G_x = g_1'G_x$. Thus the map ψ is well defined.

To show the map ψ is one to one, suppose $x_1, x_2 \in Gx$, and $\psi(x_1) = \psi(x_2)$. Then there exist $g_1, g_2 \in G$ such that $x_1 = g_1x$, $x_2 = g_2x$, and $g_2 \in g_1G_x$. Then $g_2 = g_1g$ for some $g \in G_x$, so $x_2 = g_2x = g_1(gx) = g_1x = x_1$. Thus ψ is one to one.

Finally, we show that each left coset of G_x in G is of the form $\psi(x_1)$ for some $x_1 \in Gx$. Let g_1G_x be a left coset. Then if $g_1x = x_1$, we have $g_1G_x = \psi(x_1)$. Thus ψ maps Gx one to one onto the collection of left cosets so $|Gx| = (G : G_x)$.

If $|G|$ is finite, then the equation $|G| = |G_x|(G : G_x)$ shows that $|Gx| = (G : G_x)$ is a divisor of $|G|$. ◆

16.17 Example Let X be the D_4-set in Example 16.8, with action table given by Table 16.10. With $G = D_4$, we have $G1 = \{1, 2, 3, 4\}$ and $G_1 = \{\rho_0, \delta_2\}$. Since $|G| = 8$, we have $|G1| = (G : G_1) = 4$. ▲

We should remember not only the cardinality equation in Theorem 16.16 but also that the *elements of G carrying x into $g_1 x$ are precisely the elements of the left coset $g_1 G_x$*. Namely, if $g \in G_x$, then $(g_1 g)x = g_1(gx) = g_1 x$. On the other hand, if $g_2 x = g_1 x$, then $g_1^{-1}(g_2 x) = x$ so $(g_1^{-1} g_2)x = x$. Thus $g_1^{-1} g_2 \in G_x$ so $g_2 \in g_1 G_x$.

EXERCISES 16

Computations

In Exercises 1 through 3, let

$$X = \{1,\ 2,\ 3,\ 4,\ s_1, s_2, s_3, s_4, m_1, m_2, d_1, d_2, C, P_1, P_2, P_3, P_4\}$$

be the D_4-set of Example 16.8 with action table in Table 16.10. Find the following, where $G = D_4$.

1. The fixed sets X_σ for each $\sigma \in D_4$, that is, $X_{\rho_0}, X_{\rho_1}, \cdots, X_{\delta_2}$
2. The isotropy subgroups G_x for each $x \in X$, that is, $G_1, G_2, \cdots, G_{P_3}, G_{P_4}$
3. The orbits in X under D_4

Concepts

In Exercises 4 and 5, correct the definition of the italicized term without reference to the text, if correction is needed, so that it is in a form acceptable for publication.

4. A group G *acts faithfully* on X if and only if $gx = x$ implies that $g = e$.
5. A group G is *transitive* on a G-set X if and only if, for some $g \in G$, gx can be every other x.
6. Let X be a G-set and let $S \subseteq X$. If $Gs \subseteq S$ for all $s \in S$, then S is a **sub-G-set.** Characterize a sub-G-set of a G-set X in terms of orbits in X and G.
7. Characterize a transitive G-set in terms of its orbits.
8. Mark each of the following true or false.

_____ a. Every G-set is also a group.

_____ b. Each element of a G-set is left fixed by the identity of G.

_____ c. If every element of a G-set is left fixed by the same element g of G, then g must be the identity e.

_____ d. Let X be a G-set with $x_1, x_2 \in X$ and $g \in G$. If $gx_1 = gx_2$, then $x_1 = x_2$.

_____ e. Let X be a G-set with $x \in X$ and $g_1, g_2 \in G$. If $g_1 x = g_2 x$, then $g_1 = g_2$.

_____ f. Each orbit of a G-set X is a transitive sub-G-set.

_____ g. Let X be a G-set and let $H \leq G$. Then X can be regarded in a natural way as an H-set.

_____ h. With reference to (g), the orbits in X under H are the same as the orbits in X under G.

_____ i. If X is a G-set, then each element of G acts as a permutation of X.

_____ j. Let X be a G-set and let $x \in X$. If G is finite, then $|G| = |Gx| \cdot |G_x|$.

9. Let X and Y be G-sets with the *same* group G. An **isomorphism** between G-sets X and Y is a map $\phi : X \to Y$ that is one to one, onto Y, and satisfies $g\phi(x) = \phi(gx)$ for all $x \in X$ and $g \in G$. Two G-sets are **isomorphic** if such an isomorphism between them exists. Let X be the D_4-set of Example 16.8.

a. Find two distinct orbits of X that are isomorphic sub-D_4-sets.
b. Show that the orbits $\{1, 2, 3, 4\}$ and $\{s_1, s_2, s_3, s_4\}$ are not isomorphic sub-D_4-sets. [*Hint:* Find an element of G that acts in an essentially different fashion on the two orbits.]
c. Are the orbits you gave for your answer to part (a) the only two different isomorphic sub-D_4-sets of X?

10. Let X be the D_4-set in Example 16.8.

a. Does D_4 act faithfully on X?
b. Find all orbits in X on which D_4 acts faithfully as a sub-D_4-set.

Theory

11. Let X be a G-set. Show that G acts faithfully on X if and only if no two distinct elements of G have the same action on each element of X.

12. Let X be a G-set and let $Y \subseteq X$. Let $G_Y = \{g \in G \mid gy = y \text{ for all } y \in Y\}$. Show G_Y is a subgroup of G, generalizing Theorem 16.12.

13. Let G be the additive group of real numbers. Let the action of $\theta \in G$ on the real plane $\mathbb{R}^2$ be given by rotating the plane counterclockwise about the origin through θ radians. Let P be a point other than the origin in the plane.

a. Show $\mathbb{R}^2$ is a G-set.
b. Describe geometrically the orbit containing P.
c. Find the group G_P.

Exercises 14 through 17 show how all possible G-sets, up to isomorphism (see Exercise 9), can be formed from the group G.

14. Let $\{X_i \mid i \in I\}$ be a disjoint collection of sets, so $X_i \cap X_j = \varnothing$ for $i \neq j$. Let each X_i be a G-set for the same group G.

a. Show that $\bigcup_{i\in I} X_i$ can be viewed in a natural way as a G-set, the **union** of the G-sets X_i.
b. Show that every G-set X is the union of its orbits.

15. Let X be a transitive G-set, and let $x_0 \in X$. Show that X is isomorphic (see Exercise 9) to the G-set L of all left cosets of G_{x_0}, described in Example 16.7. [*Hint:* For $x \in X$, suppose $x = gx_0$, and define $\phi : X \to L$ by $\phi(x) = gG_{x_0}$. Be sure to show ϕ is well defined!]

16. Let X_i for $i \in I$ be G-sets for the same group G, and suppose the sets X_i are not necessarily disjoint. Let $X'_i = \{(x, i) \mid x \in X_i\}$ for each $i \in I$. Then the sets X'_i are disjoint, and each can still be regarded as a G-set in an obvious way. (The elements of X_i have simply been tagged by i to distinguish them from the elements of X_j for $i \neq j$.) The G-set $\bigcup_{i\in I} X'_i$ is the **disjoint union** of the G-sets X_i. Using Exercises 14 and 15, show that every G-set is isomorphic to a disjoint union of left coset G-sets, as described in Example 16.7.

17. The preceding exercises show that every G-set X is isomorphic to a disjoint union of left coset G-sets. The question then arises whether left coset G-sets of distinct subgroups H and K of G can themselves be isomorphic. Note that the map defined in the hint of Exercise 15 depends on the choice of x_0 as "base point." If x_0 is replaced by g_0x_0 and if $G_{x_0} \neq G_{g_0x_0}$, then the collections L_H of left cosets of $H = G_{x_0}$ and L_K of left cosets of $K = G_{g_0x_0}$ form distinct G-sets that must be isomorphic, since both L_H and L_K are isomorphic to X.

a. Let X be a transitive G-set and let $x_0 \in X$ and $g_0 \in G$. If $H = G_{x_0}$ describe $K = G_{g_0x_0}$ in terms of H and g_0.
b. Based on part (a), conjecture conditions on subgroups H and K of G such that the left coset G-sets of H and K are isomorphic.
c. Prove your conjecture in part (b).

18. Up to isomorphism, how many transitive $\mathbb{Z}_4$ sets X are there? (Use the preceding exercises.) Give an example of each isomorphism type, listing an action table of each as in Table 16.10. Take lowercase names a, b, c, and so on for the elements in the set X.

19. Repeat Exercise 18 for the group $\mathbb{Z}_6$.

20. Repeat Exercise 18 for the group S_3. List the elements of S_3 in the order ι, $(1, 2, 3)$, $(1, 3, 2)$, $(2, 3)$, $(1, 3)$, $(1, 2)$.

SECTION 17 †APPLICATIONS OF G-SETS TO COUNTING

This section presents an application of our work with G-sets to counting. Suppose, for example, we wish to count how many distinguishable ways the six faces of a cube can be marked with from one to six dots to form a die. The standard die is marked so that when placed on a table with the 1 on the bottom and the 2 toward the front, the 6 is on top, the 3 on the left, the 4 on the right, and the 5 on the back. Of course, other ways of marking the cube to give a distinguishably different die are possible.

Let us distinguish between the faces of the cube for the moment and call them the bottom, top, left, right, front, and back. Then the bottom can have any one of six marks from one dot to six dots, the top any one of the five remaining marks, and so on. There are $6! = 720$ ways the cube faces can be marked in all. Some markings yield the same die as others, in the sense that one marking can be carried into another by a rotation of the marked cube. For example, if the standard die described above is rotated 90° counterclockwise as we look down on it, then 3 will be on the front face rather than 2, but it is the same die.

There are 24 possible positions of a cube on a table, for any one of six faces can be placed down, and then any one of four to the front, giving $6 \cdot 4 = 24$ possible positions. Any position can be achieved from any other by a rotation of the die. These rotations form a group G, which is isomorphic to a subgroup of S_8 (see Exercise 45 of Section 8). We let X be the 720 possible ways of marking the cube and let G act on X by rotation of the cube. We consider two markings to give the same die if one can be carried into the other under action by an element of G, that is, by rotating the cube. In other words, we consider each *orbit* in X under G to correspond to a single die, and different orbits to give different dice. The determination of the number of distinguishable dice thus leads to the question of determining the number of orbits under G in a G-set X.

The following theorem gives a tool for determining the number of orbits in a G-set X under G. Recall that for each $g \in G$ we let X_g be the set of elements of X left fixed by g, so that $X_g = \{x \in X \mid gx = x\}$. Recall also that for each $x \in X$, we let $G_x = \{g \in G \mid gx = x\}$, and Gx is the orbit of x under G.

17.1 Theorem **(Burnside's Formula)** Let G be a finite group and X a finite G-set. If r is the number of orbits in X under G, then

$$r \cdot |G| = \sum_{g \in G} |X_g|. \qquad \textbf{(1)}$$

† This section is not used in the remainder of the text.