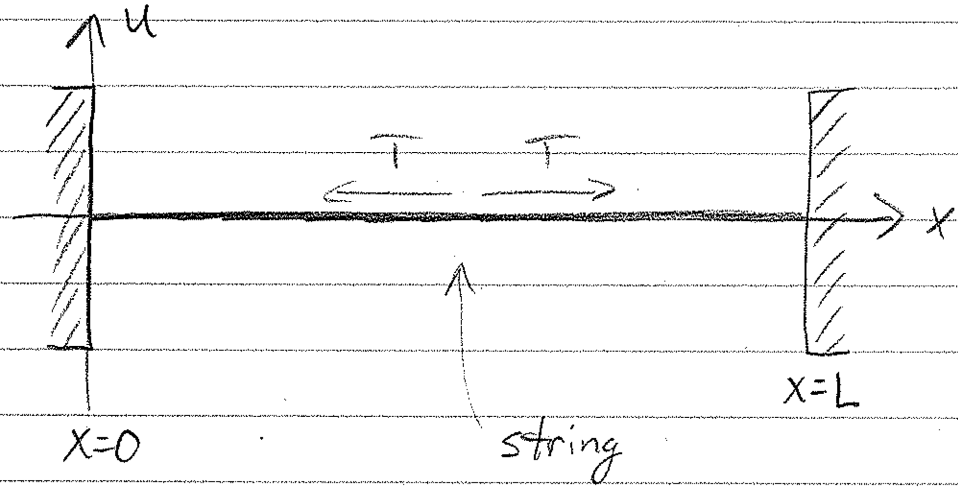
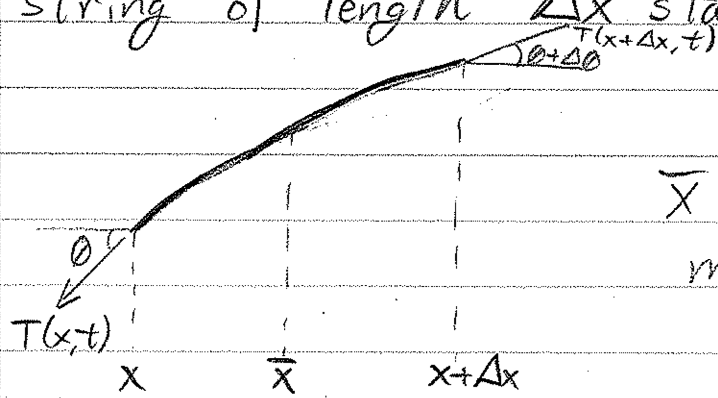
3/31) <u>The Wave Equation</u>

A derivation of the wave equation in spacial dimension for the transverse vibrations of an elastic string

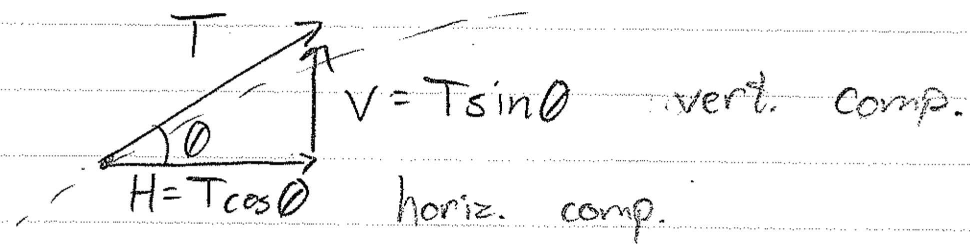


There is obviously tension in the string otherwise it would not be straight.
Now @ time $t=0$ set the string in motion. Ignoring any damping effects we will derive the equation of motion of the string. Since derivatives are "local" operations we will focus in on a small piece of the string of length Δx starting at x.



$\bar{x}$ is center of mass.

Assume that the motion of the string is small enough so that motion occurs in a vertical line (for each x). The tension in the string will be denoted by $T(x,t)$, $u(x,t)$ will be the vertical displacement of the string at the point x and time t, and ρ is mass per unit length of the string. Observe that T is always tangential.



Using Newton's Law $F = ma$ and noting that there is no horizontal acceleration the equation becomes

(1) $$T(x+\Delta x, t)\cos(\theta+\Delta\theta) - T(x,t)\cos\theta = 0$$

for the horizontal parts and

(2) $$T(x+\Delta x, t)\sin(\theta+\Delta\theta) - T(x,t)\sin\theta = \underbrace{\rho\Delta x}_{\text{mass}}\,\overbrace{u_{tt}(\bar{x},t)}^{\text{accel.}}$$

for the vertical parts. Here we have neglected the weight of the string in (2).

Rewrite (2) as:

$$V(x+\Delta x, t) - V(x,t) = \rho \Delta x\, u_{tt}(\bar{x}, t)$$

Rewrite again to get:

$$\frac{V(x+\Delta x, t) - V(x,t)}{\Delta x} = \rho u_{tt}(\bar{x}, t)$$

Take the limit as $\Delta x \to 0$, then

(3) $$V_x(x,t) = \rho u_{tt}(x,t)$$

Note that H is independent of x.

We would like to rewrite (3) completely in terms of u. So let's figure out what V is:

$$V(x,t) = H(t) \tan\theta = H(t)\, u_x(x,t)$$

Since $\tan\theta \sim \frac{\Delta u}{\Delta x}$ and we let $\Delta x \to 0$.

So (3) becomes:

$$(H u_x)_x = \rho u_{tt}$$

-or-

(4) $$H(t)\, u_{xx}(x,t) = \rho u_{tt}(x,t)$$

For small θ, $\cos\theta \approx 1$ so we may replace H with T:

$$T u_{xx} = \rho u_{tt}$$

-or-

(5) $$a^2 u_{xx} = u_{tt}$$ where

(6) $$a^2 = T/\rho$$

Eq (5) is the wave equation in one spacial dimension.

a is the wave velocity.

We can expand to higher dimensions:

2-D: $a^2(u_{xx}+u_{yy}) = u_{tt}$

3-D: $a^2(u_{xx}+u_{yy}+u_{zz}) = u_{tt}$

Fourier Series

Consider the series:

(7) $$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$$

If it converges to a function f it is said to be the Fourier series of f.

Periodic Functions

A function f is periodic with period T if

$$f(x+T) = f(x)$$

$\forall x \in D(f)$. The smallest such T is called the fundamental period of f.

The functions $\cos\frac{m\pi x}{L}$ & $\sin\frac{m\pi x}{L}$ have period $T = \frac{2L}{m}$.

Note that $\sin\frac{m\pi x}{L}$ and $\cos\frac{m\pi x}{L}$ have a common period of $2L$.

It can be shown that a convergent infinite sum of functions of period T will be a periodic function of period T. Thus the Fourier series will have a period of $2L$.

Orthogonality of Functions

Two vectors are orthogonal if $v \cdot w = 0$, but when are two functions orthogonal?

Define an inner product by:

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

where $f, g : [a,b] \longrightarrow \mathbb{R}$.

f & g are orthogonal if $\langle f, g \rangle = 0$.

A set of functions is called mutually orthogonal if each distinct pair is orthogonal.

Claim: The set $\{\cos\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}\}_{m=1}^{\infty}$ is mutually orthogonal on $[-L, L]$.

pf/ It is easy to verify that:

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx = L\,\delta_{mn}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = 0$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx = L\,\delta_{mn}$$

where $\delta_{mn} = \begin{cases} 1, & m=n \\ 0, & m \neq n \end{cases}$

Suppose we have a series of the form (7) and it converges to f: i.e

(8) $$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$$

Multiply (8) by $\cos\frac{n\pi x}{L}$ where $n \in \mathbb{N}$ is fixed and integrate w.r.t. x from $-L$ to L to get

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\, dx = \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}\, dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\, dx$$

$$+ \sum_{m=1}^{\infty}\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\, dx = 0 + L a_n + 0$$

(9) So, $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\, dx, \quad n \in \mathbb{N}$$

We also need a_0. Integrate (8) from $-L$ to L:

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} \frac{a_0}{2}\,dx + \sum_{m=1}^{\infty}\left(\int_{-L}^{L}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)dx\right)$$

$$= La_0 + 0$$

(10) $$\Rightarrow a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

There is no b_0, and b_n is calculated by multiplying (8) by $\sin\frac{m\pi x}{L}$, then

(11) $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n \in \mathbb{N}.$$

We can actually use (9) to compute (10) so to compute a Fourier series we use:

$f(x)$ has period T. $L = \frac{1}{2}T$.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n \in \mathbb{N}_0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n \in \mathbb{N}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

Examples — Triangular Wave

① $f(x) = \begin{cases} -x, & -2 \le x < 0 \\ x, & 0 \le x < 2, \end{cases} \quad f(x) = f(x+4)$

$T = 4 \Rightarrow L = 2$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} -x\,dx + \int_0^2 x\,dx\right]$$

$$= \frac{1}{2}\left(\frac{-x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_0^2\right) = \frac{1}{2}(2+2) = 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left(\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx + \int_0^2 x\cos\frac{n\pi x}{2}\,dx\right)$$

$$\overset{\text{I.B.P.}}{=} \frac{1}{2}\left(\frac{-2}{n\pi}x\sin\frac{n\pi x}{2} - \left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi x}{2}\right)\Big|_{-2}^{0}$$

$$+ \frac{1}{2}\left(\frac{2}{n\pi}x\sin\frac{n\pi x}{2} + \left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi x}{2}\right)\Big|_0^2$$

$$= \frac{4}{(n\pi)^2}(\cos n\pi - 1), \quad m \in \mathbb{N}$$

$$= \begin{cases} -8/(m\pi)^2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

You can calculate:

$$b_n = 0 \quad \forall n \in \mathbb{N}.$$

Thus,

$$f(x) = 1 - \frac{8}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}\cos\frac{n\pi x}{2} = 1 - \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{(2m-1)\pi x}{2}$$

Example 2

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}, \quad f(x) = f(x+2)$$