Throughout these notes we will frequently refer to the Sturm-Liouville problem:

$$\begin{array}{ll} -[p(x)y']' + q(x)y = \lambda r(x)y, & 0 < x < 1 \\ \alpha_1 y(0) + \alpha y'(0) = 0, & \beta_1 y(1) + \beta_2 y'(1) = 0 \end{array} \tag{1}$$

the function $r(x)$ is called the weight function of the Sturm-Liouville problem. The boundary conditions here are called separated since each condition only depend on one of the boundaries (i.e., there is nothing like $y(0)+y'(1) = 0$).

It is commonplace to define the linear homogeneous differential operator L on the vector space of $\mathcal{C}^2([0,1])$ functions by:

$$L[y] = -[p(x)y']' + q(x)y$$

This allows us to rewrite the Sturm-Liouville equation (1) as:

$$L[y] = \lambda r(x)y$$

Define the inner product of two real valued functions u, v by:

$$\langle u, v\rangle = \int_0^1 u(x)v(x); dx.$$

Then we have what is known as Lagrange's identity:

$$\langle L[u], v\rangle - \langle u, L[v]\rangle = 0 \tag{2}$$

If a linear homogeneous differential operator L satisfies (2), the operator is called Self-Adjoint.

Theorem 1. *If ϕ_1 and ϕ_2 are two eigenfunctions of the Sturm-Liouville problem* (1) *corresponding to eigenvalues λ_1 and λ_2, respectively, and if $\lambda_1 \neq \lambda_2$, then:*

$$\int_0^1 r(x)\phi_1(x)\phi_2(x)\ dx = 0.$$

This theorem says that the eigenfunctions of (1) are orthogonal with respect to the weight function $r(x)$. Another way of saying this is that they are orthogonal in the r-weighted inner product of functions:

$$\langle u, v\rangle_r = \int_0^1 r(x)u(x)v(x)\ dx.$$

Let $y_1, y_2, ..., y_n, ...$ be the eigenfunctions of (1). We say that the eigenfunction y_n is normalized with respect to the weight function $r(x)$ if:

$$\langle y_n, y_n\rangle_r = 1.$$

If each of the eigenfunctions are normalized already, we see that the set $\{y_n\}$ forms an orthonormal set with respect to the weight function $r(x)$. If the eigenfunction y_n is not normalized, to normalize it we multiply y_n by a constant k_n and demand that:

$$\langle k_n y_n, k_n y_n\rangle_r = \int_0^1 r(x)k_n^2 y_n^2\ dx = 1,$$

then use that to solve for k_n. Then the function $\phi_n = k_n y_n$ is called the normalized eigenfunction.

Since the ϕ_n (assuming that they are all normalized) above form an orthonormal set, we can attempt to expand a function f (satisfying suitable conditions) as follows. Supposing that:

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

then since the functions ϕ_n are orthonormal in the r-weighted inner product we get that:

$$c_n = \langle f(x), \phi_n(x)\rangle_r .$$