HW2: Ch 1 # 4, 6, 8

(4) Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove $\alpha \leq \beta$.

pf/ α lower bound of $E \Rightarrow \alpha \leq x \quad \forall x \in E$

β upper " " " $\Rightarrow x \leq \beta \quad \forall x \in E$

$\Rightarrow \alpha \leq x \leq \beta \quad \forall x \in E \Rightarrow \alpha \leq \beta$.

$\square$ w/ pic.

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(6) Fix $b > 1$.

(a) If $m, n, p, q \in \mathbb{Z}$, $n > 0$, $q > 0$, and $r = \frac{m}{n} = \frac{p}{q}$, prove that $(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$.

(b) Prove that $b^{r+s} = b^r b^s$ if $r, s \in \mathbb{Q}$.

(c) $x \in \mathbb{R}$, $B(x) = \{b^t \mid t \in \mathbb{Q} \text{ and } t \leq x\}$. Prove that

$b^r = \sup B(r)$

(d) Prove $b^{x+y} = b^x b^y \quad \forall x, y \in \mathbb{R}$.

pf/ ⓐ Since $\frac{m}{n} = \frac{p}{q} \Rightarrow mq = np \Rightarrow b^{mq} = b^{np}$ ②

By Theorem 1.21 $(b^{mq})^{\frac{1}{nq}} = (b^{np})^{\frac{1}{nq}} \Leftrightarrow (b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$.
(for uniqueness) ✓

ⓑ Let $r, s \in \mathbb{Q}$. Suppose $r = \frac{m}{n}$ & $s = \frac{p}{q}$. Then

$$b^{r+s} = b^{\frac{m}{n}+\frac{p}{q}} = b^{\frac{mq+np}{nq}} \overset{ⓐ}{=} (b^{mq+np})^{\frac{1}{nq}} \overset{\mathbb{R} \text{ a field}}{=} (b^{mq} b^{np})^{\frac{1}{nq}}$$

$$= \left(b^{\frac{m}{n}\cdot nq} b^{\frac{p}{q}\cdot nq}\right)^{\frac{1}{nq}} \overset{ⓐ}{=} \left[\left(b^{\frac{m}{n}} b^{\frac{p}{q}}\right)^{nq}\right]^{\frac{1}{nq}} \overset{1.21}{=} b^{\frac{m}{n}} b^{\frac{p}{q}} = b^r b^s.$$ ✓

ⓒ First note that $b^r \in B(r)$. Then for any $t \in \mathbb{Q}$ with $t \leq r$: $b^r = b^{t+(r-t)} = b^t b^{r-t} \geq b^t 1^{r-t} = b^t$.

Thus $b^r = \sup B(r)$. ✓

ⓓ First note that for any $t \leq x$ and $s \leq y$:

$b^x b^y = \sup B(x) \sup B(y) \geq b^t b^s = b^{t+s}$

Thus $b^x b^y$ is an upper bound of $B(x+y)$.

$\Rightarrow b^x b^y \geq b^{x+y}$

Fix $\varepsilon > 0$

Now by def. of sup, choose $t \leq x$ s.t. $b^x < b^t + \varepsilon$ and $s \leq y$ s.t. $b^y < b^s + \varepsilon$. Then $b^x b^y < b^t b^s + \varepsilon(b^t + b^s) + \varepsilon^2 = b^{t+s} + \varepsilon(b^t + b^s) + \varepsilon^2$

Take sup of both sides: $b^x b^y < b^{x+y} + \varepsilon(b^x + b^y) + \varepsilon^2$

ε arbitrary $\Rightarrow b^x b^y \leq b^{x+y}$

$\therefore b^x b^y = b^{x+y}$. ✓

☐ pic.

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8 Prove that no order can be defined in the complex field that turns it into an ordered field.

pf/ Since $i \neq 0$ we must have $i^2 > 0$ by 1.18d. Since $i^2 = -1$ either $-1 > 0$ or we have a contradiction. Suppose $-1 > 0 \Rightarrow -(-1) = 1 < 0$ also a contradiction. $\therefore$ $\mathbb{C}$ cannot be made into an ordered field.

<u>Optional</u> : $E_n := \{x \in \mathbb{R} \mid x \geq n\} = [n, \infty)$, $n \in \mathbb{N}$

Find $\bigcup_{n=1}^{\infty} E_n$ and $\bigcap_{n=1}^{\infty} E_n$

<u>Sol</u> : $\bigcup_{n=1}^{\infty} E_n = [1, \infty)$ since $E_1 \supset E_2 \supset E_3 \supset \cdots$

$\bigcap_{n=1}^{\infty} E_n = \emptyset$ since $\forall x \in \mathbb{R}\ \exists n \in \mathbb{N}$ s.t. $n > x$.