Homework 3-a, for 10/8, 11, . Math 151A, fall 2010. Deadline 10/20

(1) Show that a k-cell is convex. (the definition of k-cell is Def.2.17, p31.)

(2)Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Do the problems 4, 8, 9 , 10 , 11-(d5) in page 43-44 of Rudin.

HW 3-a

① Show that a k-cell is convex.

pf/ Let $K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$ be a k-cell.

Let $\vec{x} = (x_1, \cdots, x_k)$ and $\vec{y} = (y_1, \cdots, y_k)$ be in K, $x \neq y$.

WTS $t\vec{x} + (1-t)\vec{y} \in K \ \forall t \in [0,1]$

It suffices to show $tx_i + (1-t)y_i \in [a_i, b_i] \ \forall t \in [0,1] \ \forall i$.

Define $z_i = \frac{a_i + b_i}{2}$ and $r_i = \frac{b_i - a_i}{2}$. Then $\forall \eta \in [a_i, b_i]$,

$|\eta - z| \leq r_i$, in particular $|x_i - z_i| \leq r_i$ and $|y_i - z_i| \leq r_i$.

Thus $|tx_i + (1-t)y_i - z_i| = |tx_i + (1-t)y_i - [t + (1-t)]z_i|$

$= |t(x_i - z_i) + (1-t)(y_i - z_i)| \leq t|x_i - z_i| + (1-t)|y_i - z_i|$

$\leq tr_i + (1-t)r_i = r_i$

$\therefore tx_i + (1-t)y_i \in [a_i, b_i] \ \forall t \in [0,1]$

Since i was arbitrary, $t\vec{x} + (1-t)\vec{y} \in K \ \forall t \in [0,1]$

$\therefore K$ is convex since $\vec{x}$ and $\vec{y}$ were arbitrary.

 pic.

② Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.

pf/ Claim A subset D of a topological space X is dense if every nonempty open set $\mathcal{U} \subset X$ satisfies $\mathcal{U} \cap D \neq \emptyset$.

(pf/ Exercise.)

Let $\mathcal{U} \subset \mathbb{R}$ be open and nonempty. Then for any $x \in \mathcal{U}$ $\exists\, N_\varepsilon(x) \subset \mathcal{U}$ a nbhd of x. Since $x + \frac{\varepsilon}{2} \in N_\varepsilon(x)$ and we know $\exists q \in \mathbb{Q}$ s.t. $x < q < x + \frac{\varepsilon}{2}$, $\mathbb{Q} \cap N_\varepsilon(x) \neq \emptyset$

And since $N_\varepsilon(x) \subset \mathcal{U} \Rightarrow \mathbb{Q} \cap \mathcal{U} \neq \emptyset$.

$\therefore$ $\mathbb{Q}$ is dense in $\mathbb{R}$.

Ch 2: #4, 8, 9, 10, 11 - (d_5)

(4) Is the set of all irrational real numbers countable?

Ans/ Theorem 2.14 (2.43) $\Rightarrow \mathbb{R}$ is uncountable.

Since $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ ($\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$) and $\mathbb{Q}$ is countable, it follows that $\mathbb{I}$ is uncountable.

(8) (i) Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? (ii) Answer the same question for closed sets in $\mathbb{R}^2$.

Ans/ (i) Yes. Let $x \in E$. Since E is open $\exists N_r(x) \subset E$, Let $N_\varepsilon(x)$ be any nbhd of x. Then either (1) $N_\varepsilon(x) \subset N_r(x)$ or (2) $N_r(x) \subset N_\varepsilon(x)$. Both imply $\exists y \in N_\varepsilon(x) \cap E$, $y \neq x$.

$\therefore x$ is a limit point of E.

(ii) No. Let $E = \{x\}$. E is closed and for every nbhd $N_r(x)$ of x, $E \cap (N_r(x) \setminus \{x\}) = \emptyset$.

$\therefore x$ is not a limit point of E.

⑨ Let E° denote the set of all interior points of a set E.

ⓐ Prove that E° is always open.

ⓑ Prove that E is open iff $E^\circ = E$

ⓒ If $G \subset E$ and G is open, prove $G \subset E^\circ$.

ⓓ Prove that the complement of E° is the closure of the complement of E.

ⓔ Do E and $\bar{E}$ always have the same interiors?

ⓕ Do E and E° always have the same closures?

Pf/ ⓐ Let $x \in E^\circ$. $\Rightarrow \exists N(x) \subset E$. Claim: $E^\circ = \bigcup_{x \in E^\circ} N(x)$

Clearly $E^\circ \subset \bigcup_{x \in E^\circ} N(x)$. Let $y \in \bigcup_{x \in E^\circ} N(x)$. Then $y \in N(x) \subset E$ for some $x \in E^\circ$. Thus y is an interior point of E. Thus $y \in E^\circ$. $\therefore E^\circ = \bigcup_{x \in E^\circ} N(x)$. Since the union of open sets is open, E° is open.

ⓑ ($\Leftarrow$) Obvious since E° is open.

($\Rightarrow$) If E is open $\Rightarrow$ every point of E is an interior point of E $\Rightarrow E \subset E^\circ$, Since $E^\circ \subset E$, $E = E^\circ$.

(c) G open $\Rightarrow G = G^\circ$. Thus for each $x \in G$, $\exists N(x) \subset G$.
By proof in (a) $G^\circ = \bigcup_{x \in G^\circ} N(x)$. Since $G \subset E$, $\forall x \in G$
$N(x) \subset G \subset E$, thus $G^\circ = G \subset E^\circ$, since $x \in E^\circ$.

(d) <u>WTS</u> $(E^\circ)^c = \overline{E^c}$.

($\subset$) Let $x \in (E^\circ)^c \Rightarrow x \notin E^\circ \Rightarrow \forall \mathcal{U}$ nbhd of x, $\mathcal{U} \not\subset E$
$\Rightarrow \forall \mathcal{U}$ nbhd of x $\mathcal{U} \cap E^c \neq \emptyset \Rightarrow x \in \overline{E^c}$.

($\supset$) Let $x \in \overline{E^c} \Rightarrow x \in E^c \cup (E^c)'$

If $x \in E^c \Rightarrow x \notin E \Rightarrow x \notin E^\circ \subset E$.

If $x \in (E^c)' \Rightarrow \forall \mathcal{U}$ nbhd of x, $\mathcal{U} \cap E^c \neq \emptyset$
$\Rightarrow \forall$ nbhd $\mathcal{U}$ of x $\mathcal{U} \not\subset E$
$\Rightarrow x \notin E^\circ \Rightarrow x \in (E^\circ)^c$.

$\therefore (E^\circ)^c = \overline{E^c}$.

(e) No. Let $E = (0,1) \cup (1,2)$. Then $E^\circ = E$ and $\overline{E} = [0,2]$.
But $(\overline{E})^\circ = (0,2)$ $\therefore E^\circ \neq (\overline{E})^\circ$.

(f) No. Let $E = \{0\}$. Then $E^\circ = \emptyset$ and $\overline{E} = E$, while
$\overline{E^\circ} = \emptyset$.

(10) Let X be an infinite set. For $p \in X$ and $q \in X$, define $d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$. Prove d is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Ans/ (i) Suppose $p \neq q \in X$. Then $d(p,q) = 1 > 0$.
Also $d(p,p) = 0$ since $p = p$.

(ii) Since $p = q \Rightarrow q = p$ & $p \neq q \Rightarrow q \neq p$ we have $d(p,q) = d(q,p)$.

(iii) Let $p, q, r \in X$. Then:

$$d(p,q) = 1 \Rightarrow p \neq q \Rightarrow d(p,r) = 1 \text{ or } d(r,q) = 1$$
$$\Rightarrow d(p,q) \leq d(p,r) + d(r,q).$$

$d(p,q) = 0 \Rightarrow p = q$. Since $d(p,r) = 0$ or 1 and $d(r,q) = 0$ or 1

$$0 = d(p,q) \leq \min_{r \in X} \{ d(p,r) + d(r,q) \} = 0$$

$\therefore d$ is a metric.

Let $p \in X$. Then the open nbhd $N_{\frac{1}{2}}(p) = \{p\}$ so that all single point sets in X are open. It follows that every subset of X is open. Since every set in X is the complement of some other set (namely $E = (E^c)^c$), every set in X is closed. The compact sets are the finite subsets since the covering of an infinite set by singletons has no finite subcover.

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(11) For $x, y \in \mathbb{R}$, define $d_5(x,y) = \dfrac{|x-y|}{1+|x-y|}$.

Determine whether d_5 is a metric.

Pf/ (i) Clearly $d_5(x,y) \geq 0$. Suppose $d_5(x,y) = 0$.

$$\Rightarrow \frac{|x-y|}{1+|x-y|} = 0 \Rightarrow |x-y| = 0 \Rightarrow x = y.$$

(ii) Since $|x-y| = |y-x|$, $d_5(x,y) = d_5(y,x)$.

(iii) Let $x, y, z \in \mathbb{R}$. We need to show $\dfrac{|x-y|}{1+|x-y|} \leq \dfrac{|x-z|}{1+|x-z|} + \dfrac{|z-y|}{1+|z-y|}$

Equivalently: $$\frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} - \frac{|x-y|}{1+|x-y|} \geq 0.$$

$$\frac{|x-z|}{1+|x-z|}+\frac{|z-y|}{1+|z-y|}-\frac{|x-y|}{1+|x-y|}$$

$$=\frac{\left(\begin{array}{l}|x-z|+\cancel{|x-z||x-y|}+|x-z||z-y|+\cancel{|x-z||x-y||z-y|}+|z-y|+|z-y||x-z|+\cancel{|z-y||x-y|}\\ \quad +|z-y||x-z||x-y|-|x-y|-\cancel{|x-y||x-z|}-\cancel{|x-y||z-y|}-\cancel{|x-y||x-z||z-y|}\end{array}\right)}{(1+|x-z|)(1+|z-y|)(1+|x-y|)}$$

$$=\frac{\overbrace{(|x-z|+|z-y|-|x-y|)}^{\geq 0 \text{ by } (\star)}+\overbrace{|x-z||z-y|}^{\geq 0}+\overbrace{|z-y||x-z|}^{\geq 0}+\overbrace{|z-y||x-z||x-y|}^{\geq 0}}{(1+|x-z|)(1+|z-y|)(1+|x-y|)}\geq 0$$

since $|x-y|\leq|x-z|+|z-y|$ $\star$

$\therefore$ d_5 is a metric.

HW 3-b Ch. 2 #12, 14

(12) Let $K \subset \mathbb{R}$ consist of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition.

pf/ Let $\mathcal{O} = \{U_i\}_{i \in I}$, I an index set, be an open cover of K. Since 0 is a limit point of K, by theorem 2.20, every neighborhood of 0 contains infinitely many points of K. Since, by the structure of K, there are only finitely many points of K not contained in any given neighborhood of 0, we construct the finite subcover as follows: Let V_0 be any U_i containing 0. Let $x_1, \ldots, x_n$ be the finitely many points of K missed by V_0. For each x_j $\exists\, U_{i_j} \in \mathcal{O}$ s.t. $x_j \in U_{i_j}$. Let $V_j = U_{i_j}$.

Then $\mathcal{F} = \{V_j\}_{j=1}^{n}$ is a finite subcover of $\mathcal{O}$.

$\therefore$ K is compact. $\blacksquare$

(14) Give an example of an open cover of $(0,1)$ which has no finite subcover.

Ex: The cover $\mathcal{O} = \{(\frac{1}{n}, 1)\}$ has no finite subcover.