HW 4: Ch. 2 #16

(16) Regard $\mathbb{Q}$ as a metric space with $d(p,q)=|p-q|$. Let $E=\{p\in\mathbb{Q} \mid 2<p^2<3\}$. Show E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

pf/ This topology on $\mathbb{Q}$ is induced by the usual topology on $\mathbb{R}$. $E=(-\sqrt{3},-\sqrt{2})\cup(\sqrt{2},\sqrt{3})$. Let $x\in E$, <u>WLOG</u> assume $x\in(\sqrt{2},\sqrt{3})$. Let $\alpha=d_{\mathbb{R}}(x,\sqrt{2})$ and $\beta=d_{\mathbb{R}}(x,\sqrt{3})$. Let $\eta=\min(\alpha,\beta)$. Then $B_{\frac{\eta}{2}}(x)\subset E$, so E is open. Let $\mathcal{U}$ be any neighborhood of x, then $\mathcal{U}\cap E\neq\emptyset$ is open, hence $\exists B_r(x)\subset \mathcal{U}\cap E$, $r\in\mathbb{Q}$. $\exists\, y\in B_r(x)\setminus\{x\}$ (i.e. $y=x+\frac{r}{2}$) thus x is a limit point of E. $\therefore$ E is closed. (*)

E is clearly bounded since $E\subset B_2(0)$.

Let $\mathcal{C}=\{(-2,\sqrt{3}-\frac{1}{n}) \mid n\in\mathbb{N}\}$ be an open cover of E. Then $\forall$ finite subcovers of $\mathcal{C}$, $\exists$ a largest element $(-2,\sqrt{3}-\frac{1}{N})$. Since $\sqrt{3}-\frac{1}{N}<\sqrt{3}$ $\forall N$, $\exists r\in\mathbb{Q}$ s.t. $\sqrt{3}-\frac{1}{N}<r<\sqrt{3}$ $\Rightarrow r\in E$ but r is not covered. Thus

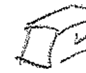
(*) This is not sufficient to show closed. 1'

We also need this:

Suppose $x \notin E$. WLOG: assume $x > \sqrt{3}$.

Let $\delta = d_{\mathbb{R}}(x, \sqrt{3})$, then $B_{\frac{\delta}{2}}(x) \cap E = \emptyset$.

$\Rightarrow E^c$ is open $\Rightarrow E$ is closed.

E is not compact since this open cover does not have a finite subcover.  pic.

Show $[0,1]$ is compact directly from the definition.

pf/ Let $\delta = 1$. Then $|x-y| \leq 1$ if $x, y \in [0,1]$. Suppose $\{G_\alpha\}$ is an open cover of $[0,1]$ with no finite subcover. Then at least one of $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ cannot be covered by a finite subcollection of $\{G_\alpha\}$, call it I_1. Repeat this with I_1 and so on. Then we get a sequence $\{I_k\}$ s.t.:

a) $I \supset I_1 \supset I_2 \supset \cdots$

b) I_k is not covered by any finite subcollection of $\{G_\alpha\}$ $\forall k$.

c) if $x, y \in I_k$, then $|x-y| \leq 2^{-k}$.

Theorem 2.39 $\Rightarrow \exists z \in \bigcap_{i=1}^{\infty} I_k$. For some α, $z \in G_\alpha$, thus since G_α is open $\exists B_r(z) \subset G_\alpha$. Choose n large enough so that $2^{-n} < r$, then by c $I_n \subset G_\alpha$, which contradicts b. $\therefore [0,1]$ is compact.