HWS: Ch. 2 # 17, 22, 29

⑰ Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

pf/ Not countable: Assume E is and write $E = \{x_n\}_1^\infty$, where each $x_n = 0.x_{n_1}x_{n_2}x_{n_3}\cdots$ where $x_{n_i} \in \{4,7\}\ \forall n, i$.

Define $y = 0.y_1y_2y_3\cdots$ where $y_i = \begin{cases} 4 & \text{if } x_{i_i} = 7 \\ 7 & \text{if } x_{i_i} = 4 \end{cases}$.

Then $y \in E$ but $y \neq x_n\ \forall n$. Thus E is not countable.

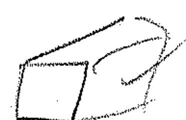
Not dense: Since $\inf_{x \in E} d(0,x) = 0.4$, E cannot be dense in E.

Compact: Clearly E is bounded as $E \subset [0,1]$. Let $p \in E$. Write $p = 0.p_1p_2p_3\cdots$, $p_i \in \{4,7\}$. Let $q_i = 0.p_1\cdots p_i$ and define $r_i = 2|p - q_i|$. Then $q_i \in B_{r_i}(p)$, and $B_{r_i}(p) \supset B_{r_{i+1}}(p)\ \forall i$. Let $\mathcal{U}$ be an open nbhd of p, then since $\mathcal{U}$ is open $\exists B_{r_i}(p) \subset \mathcal{U}$. Thus $q_i \in \mathcal{U}$, and since

$q_i \neq p \ \forall i$, we arrive at p being a limit point of E. Thus every point of E is a limit point of E.

Let $x \notin E$. Then $x = 0.a_1a_2\cdots$ with $a_i \neq 4,7$ for at least one i. If $a_i \in \{0,1,2,9\}$ the nbhd $N_{10^{-i}}(x) \subset E^c$ so $x \notin E'$, if $a_i \in \{3,5,6,8\}$ let a_j be the first term s.t. $a_j \neq 0$. Then $N_{10^{-j}}(x) \subset E^c$. $\therefore$ E is closed.

$\therefore$ E is compact & perfect.



pic

---

(22) A metric space is called <u>separable</u> if it contains a countable dense subset. Show $\mathbb{R}^k$ is separable.

pf/ <u>Claim</u>: $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$.

Let $\mathcal{U} \subset \mathbb{R}^k$ be open. Since $\mathcal{U}$ is open $\exists B_r(x) \subset \mathcal{U}$ for some $x \in \mathcal{U}$, $r > 0$. If $x \in \mathbb{Q}^k$ we are done, if not write $x = (x_1, \cdots, x_k)$. Let $r_1, \cdots, r_k > 0$ be s.t. $\sum_1^k r_i^2 < r^2$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, $\forall i$ we can find $q_i \in \mathbb{Q}$ s.t. $x_i < q_i < x_i + r_i$. Let $q = (q_1, \cdots, q_k)$. Then $\|q - x\|^2 = (q_1 - x_1)^2 + \cdots + (q_k - x_k)^2 < r_1^2 + \cdots + r_k^2 < r^2$

Thus $q \in \mathcal{U} \cap \mathbb{Q}^k$. $\therefore \mathbb{Q}^k$ is dense in $\mathbb{R}^k$. $\square$ pic

---

29 Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.

pf/ Let $\mathcal{U} \subset \mathbb{R}$ be open. Let $x \in \mathcal{U}$. $\exists y > x$ s.t. $(x,y) \subset \mathcal{U}$ and $\exists z < x$ s.t. $(z,x) \subset \mathcal{U}$. Define a_x & b_x by: $a_x = \inf\{z \mid (z,x) \subset \mathcal{U}\}$ & $b_x = \sup\{y \mid (x,y) \subset \mathcal{U}\}$.

Then $x \in I_x := (a_x, b_x)$. <u>Claim</u>: ① $I_x \subset \mathcal{U}$ & ② $a_x, b_x \notin \mathcal{U}$.

① Let $w \in I_x$. <u>WLOG</u> assume $x < w < b_x$. Then by def of b_x $\exists y > w$ s.t. $(x,y) \subset \mathcal{U}$, thus $w \in \mathcal{U}$.
$\therefore I_x \subset \mathcal{U}$.

② Suppose $b_x \in \mathcal{U}$. Since $\mathcal{U}$ is open $\exists r > 0$ s.t. $B_r(b_x) = (b_x - r, b_x + r) \subset \mathcal{U}$. Thus $(x, b_x + r) \subset \mathcal{U}$ which is a contradiction to the def. of b_x. Thus $b_x \notin \mathcal{U}$.
Similarly $a_x \notin \mathcal{U}$.

<u>Claim</u>: $\{I_x\}_{x \in \mathcal{U}}$ is a countable disjoint collection of open intervals such that $\mathcal{U} = \bigcup_{x \in \mathcal{U}} I_x$.

First note that if $x \neq y$ and $y \in I_x$, then $I_x = I_y$. So the collection is disjoint. Clearly $\bigcup_{x \in \mathcal{U}} I_x = \mathcal{U}$ since each $x \in \mathcal{U}$ is contained in its I_x.
By density of $\mathbb{Q}$ in $\mathbb{R}$, each I_x contains a rational number, thus we have a correspondence between $\{I_x\}_{x \in \mathcal{U}}$ and a subset of $\mathbb{Q}$.

$\therefore \{I_x\}_{x \in \mathcal{U}}$ is countable.

$\square$ pic